PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Two Timescale Harmonic Balance. I. Application to Autonomous One-Dimensional Nonlinear Oscillators

J. L. Summers and M. D. Savage

Phil. Trans. R. Soc. Lond. A 1992 **340**, 473-501
doi: 10.1098/rsta.1992.0077

Two timescale harmonic balance. I. Application to autonomous one-dimensional nonlinear oscillators

By J. L. Summers and M. D. Savage
Department of Applied Mathematical Studies, University of Leeds, Leeds LS2 9JT, U.K.

Contents

Two timescale harmonic balance is a semi-analytical/numerical method for deriving periodic solutions and their stability to a class of nonlinear autonomous and forced oscillator equations of the form $\ddot{x}+x=f(x,\dot{x},\lambda)$ and $\ddot{x}+x=f(x,\dot{x},\lambda,t)$, where λ is a control parameter. The method incorporates salient features from both the method of harmonic balance and multiple scales, and yet does not require an explicit small parameter.

Essentially periodic solutions are formally derived on the basis of a single assumption: 'that an N harmonic, truncated, Fourier series and its first two derivatives can represent $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ respectively'. By seeking $x(t)$ as a series of superharmonics, subharmonics, and ultrasubharmonics it is found that the method works over a wide range of parameter space provided the above assumption holds which, in practice, imposes some 'problem dependent' restriction on the magnitude of the nonlinearities. Two timescales, associated with the amplitude and phase variations respectively, are introduced by means of an implicit parameter ϵ. These timescales permit the construction of a set of amplitude evolution equations together with a corresponding stability criterion.

In Part I the method is formulated and applied to three autonomous equations, the van der Pol equation, the modified van der Pol equation, and the van der Pol equation with escape. In this case an expansion in superharmonics is sufficient to reveal Hopf, saddle node and homoclinic bifurcations which are compared with results obtained by numerical integration of the equations. In Part II the method is applied to forced nonlinear oscillators in which the solution for $x(t)$ includes

Phil. Trans. R. Soc. Lond. A (1992) **340**, 473–501
Printed in Great Britain

superharmonics, subharmonics, and ultrasubharmonics. The features of period doubling, symmetry breaking, phase locking and the Feigenbaum transition to chaos are examined.

1. Introduction

As the name suggests, two timescale harmonic balance incorporates important features from the method of multiple scales and that of harmonic balance. In attempting to convey how this combination came about it is perhaps useful to trace the development of both multiple scales and harmonic balance from the set of perturbation/averaging methods which produce periodic solutions to nonlinear equations.

In 1748 Euler proposed an initial theory of perturbation in his memoirs on the mutual perturbations of the massive planets. Attempts to analyse the nonlinear problem of perturbed two body motion can also be seen in the works of Clairaut and D'Alembert, where the problem of secular terms (or secular variations) was also encountered. Laplace and Lagrange, towards the end of the 18th century, derived approximate solutions to many celestial problems by using perturbation theory and, in the study of the Sun–Jupiter–Saturn configuration by Laplace, there are elements of the method of averaging and higher-order perturbation techniques. However, Poisson (around 1830) initiated the idea of seeking solutions in the form of a power series in terms of a small parameter. A detailed account of the history of early perturbation theory is given by Wilson (1980).

Modern perturbation methods emerged with the work of Lindstedt (1883) and Poincaré (1892) in deriving periodic solutions to general weakly nonlinear dynamical systems of the form

$$\ddot{x}+x = \mu f(x,\dot{x}), \quad \mu \ll 1. \tag{1.1}$$

Since μ is assumed to be a small parameter, the solution does not differ greatly from the solution of the harmonic equation. The problem, however, is with frequency ω which, although close to unity, is unknown and μ dependent. The essence of the Lindstedt–Poincaré technique is to expand both x and ω as a power series in μ and determine the terms in the expansion for ω by the suppression of secular terms.

A significant development in modern perturbation methods came with the idea of introducing multiple timelike variables as discussed independently by Kevorkian (1961, 1966), Cochran (1962) and Mahony (1962) and later by Nayfeh (1973). By introducing a transformation of time, t, and expanding as a power series in μ

$$\begin{aligned} t = \omega^{-1}\tau &= (1+\mu\omega_1+\mu^2\omega_2+\ldots)^{-1}\tau \\ &= \tau-\mu\omega_1\tau+O(\mu^2) \end{aligned} \tag{1.2}$$

uniformly valid expansions can be derived by selecting the ω_i so that secular terms are suppressed. The solution expansion obtained by this and the Lindstedt–Poincaré Technique reveals that the functional dependence of x on t and μ is not disjoint as x depends on the combination μt in addition to t and μ separately. Therefore x is written in the form $x = x(t,\mu t,\mu)$. Furthermore, when the expansion is taken to higher orders then

$$\begin{aligned} &x(t,\mu) = x(t,\mu t,\mu^2 t,\mu^3 t,\ldots,\mu), \text{ or equivalently} \\ &x(t,\mu) = x(T_0,T_1,T_2,T_3,\ldots,\mu), \end{aligned} \tag{1.3}$$

where $T_i(T_i = \mu^i t)$ represent different timescales. This is the basis of the method of multiple scales, which has proved a powerful tool for yielding small amplitude periodic solutions in that region of parameter space where μ is small. In fact multiple scales has applications far beyond time-dependent problems where the only requirement is that they should contain a small parameter so that uniformly valid, power series, solutions for the dependent variable can be derived.

Kuzmak (1959) considered strongly nonlinear oscillators with weak damping of the form

$$\ddot{x} + \hat{\mu} h(x, \hat{\mu} t)\, \dot{x} + g(x, \hat{\mu} t) = 0,$$

where $0 < \hat{\mu} \ll 1$ and g is nonlinear in x. He developed a two timescale approach seeking a solution in the form

$$x(\omega, \hat{\mu} t) = x_0(\omega, \hat{\mu} t) + \hat{\mu} x_1(\omega, \hat{\mu} t)$$

and equating $O(1)$ terms to derive a 'standard' equation for x_0, i.e.

$$[\mathrm{d}\omega(\tau)/\mathrm{d}t]^2\, \partial^2 x_0/\partial\omega^2 + g(x_0, \tau) = 0 \quad \text{where } \tau = \hat{\mu} t.$$

An equation for x_1 is also derived by equating $O(\hat{\mu})$ terms.

Averaging methods for the analysis of weakly nonlinear dynamical systems, as given by equation (1.1), formally began with the work of van der Pol (1927). He proposed a method of slowly varying coefficients, based upon the first harmonic in the Fourier series expansion for the solution $x(t)$. Bogoliubov & Krylov (1937) extended this idea to develop an asymptotic method in which a solution to equation (1.1) with slowly varying amplitude is sought as a power series in μ;

$$x = a \cos\phi + \mu x_1(a, \phi) + \ldots,$$

where
$$\dot{a} = \mu A_1(a) + \ldots, \quad \dot{\phi} = 1 + \mu \Omega_1(a) + \ldots, \tag{1.4}$$

and
$$\left.\begin{aligned} A_1(a) &= -\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\phi, -a\sin\phi)\sin\phi\,\mathrm{d}\phi, \\ \Omega_1(a) &= \frac{1}{2\pi a}\int_0^{2\pi} f(a\cos\phi, -a\sin\phi)\cos\phi\,\mathrm{d}\phi. \end{aligned}\right\} \tag{1.5}$$

Simultaneously Bogoliubov & Krylov (1934) developed the method of averaging for non-autonomous, weakly nonlinear equations of the form

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}, t), \tag{1.6}$$

where μ is a small parameter and f is periodic in time. By means of a simple transformation this equation is expressed in the normal form for averaging

$$\dot{y} = \mu g(y, t); \quad y \in \mathbb{R}^2 \tag{1.7}$$

and an average of g is taken over one period so as to generate an autonomous system whose stationary solutions are now of interest. The theory has been well developed (see Hale 1969; Sanders & Verhulst 1985; Lochak & Meunier 1988).

Morrison (1966) and Perko (1969) demonstrated that by using two timescales, t and τ, the first term in the expansion of the solution via multiple scales is equivalent to averaging, implying validity of the approximations on a timescale of order $1/\mu$.

Alternative ideas for analysing weakly nonlinear equations continued to develop in parallel. Bogoliubov & Krylov (1937) proposed a method known as equivalent linearization in which a harmonic solution of the form $x = a\cos\phi$ is specified with a and ϕ given by (1.4). The key idea is to derive an equivalent linearized differential equation of the form

$$\ddot{x}+\bar{\lambda}\dot{x}+\bar{\omega}^2x = O(\mu^2), \tag{1.8}$$

which, to order μ^2, has the same solution as the original equation and

$$\bar{\lambda} = \frac{\mu}{\pi a}\int_0^{2\pi} f(a\cos\phi,\, -a\sin\phi)\sin\phi\,\mathrm{d}\phi,$$

$$\bar{\omega}^2 = 1+\frac{\mu}{\pi a}\int_0^{2\pi} f(a\cos\phi,\, -a\sin\phi)\cos\phi\,\mathrm{d}\phi.$$

With $\bar{\lambda}$ and $\bar{\omega}$ so defined Bogoliubov & Krylov established two properties; namely the principle of the equivalent balance of energy and the principle of harmonic balance. In the Soviet Union these ideas were then extended by Theodorchik (1948) with a method of energy balancing and by Goldfarb (1947) with a further development of harmonic balancing. (The work of Bogoliubov & Krylov can also be found in the book of Bogoliubov & Mitropolsky (1958).) Elsewhere this linearized approach gave rise to the describing function, developed independently and from different points of view, in the works of Kochenburger (1950) in the United States, Oppelt (1947) and Tustin (1947) in the United Kingdom and in France by Loeb (1951) and Blaquière (1951). The theoretical basis for the describing function analysis is rooted in the van der Pol method of slowly varying coefficients in the sense that a nonlinear equation is reduced to a quasi-linear equation whose terms have coefficients varying slowly with time. (Detailed accounts of many of the techniques described can be found in books by Blaquière (1966) and Šiljak (1969).) In §2 of Kuzmak (1959), examples are given when the solution of the 'standard' equation is expressible as Jacobi elliptic functions. However, in §3, when the solution of the 'standard ' equation cannot be expressed in terms of special functions it is written as a Fourier series with coefficients varying on a slow timescale. The evaluation of the coefficients of the Fourier series is essentially harmonic balance.

It is from this background that modern harmonic balance emerged in which a solution to a nonlinear equation (which may or may not contain a small parameter μ) is sought as a truncated Fourier series and individual harmonics are balanced. Mickens (1984, 1986) applied the method to a class of 'antisymmetric dynamical systems', whose solutions are expressed in terms of odd harmonics only. For equations with strong nonlinearities Bejarano & Yuste (1986); Margallo & Bejarano (1987) have shown that the more harmonics included in the solution then closer is the agreement with results obtained by using numerical techniques. The one major drawback of harmonic balance is the absence of a natural stability criterion; one has to appeal to various approximate devices to test the stability of a solution.

Therefore, by the mid-1980s the point is reached where two quite different methods for deriving periodic solutions have emerged, each with its own strengths and limitations. Multiple scales is an analytical method yielding uniformly valid solutions, and their stability, but only in a restricted region of parameter space, where a given parameter μ is assumed to be small. Harmonic balance considers the periodic solution as a finite series of harmonics and can in principle yield solutions, but not their stability, over a wide range of parameter space which imposes no restriction on any parameter being small. Clearly the next step (Savage 1986) was to incorporate a two timescale approach with harmonic balance as a device for generating a stability criterion. This method, now referred to as two timescale harmonic balance (2THB), has been substantially refined so that periodic solutions can be formally derived for a class of oscillator equations on the basis of a single assumption: 'that an N harmonic, truncated, Fourier series and its first two derivatives can represent $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ respectively'. This means that there should be no discontinuities in x, $\dot{x}$ and $\ddot{x}$ and each series should contain 'sufficient harmonics'. In practice there is a limit to the number of harmonics that can be conveniently handled via symbolic manipulation and hence this imposes a restriction on the magnitude of the nonlinearities which is found to vary according to the individual problem. Applications of the method to both autonomous and forced nonlinear systems form parts I and II of this paper.

In part I autonomous nonlinear oscillators, involving a parameter, μ are considered. In each case periodic solutions are expressed as a series of superharmonics only and this is sufficient to determine Hopf, saddle node and homoclinic bifurcations and 'weak' relaxation oscillations. It is also shown that, when μ is small, identical results can be derived to those obtained by multiple scales, taken to order μ, provided two harmonics are included for asymmetric oscillators and one for antisymmetric.

In part II forced nonlinear oscillators of the form

$$\ddot{x}+\delta\dot{x}+\alpha x+\beta x^2+\gamma x^3 = F\cos\omega t$$

are considered for α, β, γ, δ up to $O(1)$. Periodic solutions are now expressed as a series of superharmonics, subharmonics and ultrasubharmonics. This permits various features of period doubling and the Feigenbaum transition to chaos, subharmonic and ultrasubharmonic resonances, and phase locking to be determined.

2. Formulation of the method

The method of 2THB is to be applied to autonomous equations which may or may not contain a small parameter. With the van der Pol equation, for example,

$$\ddot{x}+(x^2-\mu)\,\dot{x}+x = 0 \tag{2.1}$$

self-excited periodic solutions (limit cycles) will be sought both when the parameter μ is small and of order unity including the cases $\mu = 3.0$ and 5.0 corresponding to 'weak' relaxation oscillations. For μ large the work of Dorodnicyn (1947) and Carrier (1953) using asymptotic power series is relevant.

Note that equation (2.1) could be recast in the form of (1.1) by means of the transformation $x = \mu^{\frac{1}{2}}z$ so that μ is a measure of the nonlinear (damping) terms.

There are two main features of the method,

1. The equation under investigation is written in the form of a nonlinear oscillator

$$\ddot{x}+\omega^2 x = (\omega^2-1)\,x+f(x,\dot{x}), \tag{2.2}$$

where the frequency ω is initially unknown and $f(x,\dot{x})$ is a nonlinear function involving x and $\dot{x}$.

Any $2\pi/\omega$ periodic solution of (2.2) will have a Fourier expansion, and our basic assumption is that N harmonics will provide a sufficiently accurate representation. Therefore we write

$$x_{\mathrm{L}}(t) = Z_{\mathrm{L}}+\sum_{k=1}^{N}\left(A_{(k)\,\mathrm{L}}\cos(k\omega t)+B_{(k)\,\mathrm{L}}\sin(k\omega t)\right), \tag{2.3}$$

where the coefficients Z_L, $A_{(k)L}$ and $B_{(k)L}$ have subscript L to indicate that (2.3) is a solution for the limit cycle solution. In addition, $B_{(1)L}$ is set to zero without loss of generality, to give an equation for the frequency, ω, and Z_L, ω, $A_{(1)L}$, $A_{(k)L}$, and $B_{(k)L}$, $k=[2,N]$ are determined by balancing harmonics. Furthermore it is assumed that $\dot{x}(t)$ and $\ddot{x}(t)$ can be represented by the first two derivatives of (2.3) so that on substitution into equation (2.2) a Fourier series is obtained in which the Fourier coefficients are identically zero.

$$\ddot{x}+\omega^2 x-(\omega^2-1)\,x-f(x,\dot{x}) = \rho(x,\dot{x},\ddot{x},t) = \tfrac{1}{2}a_0+\sum_{k=1}^{N}(a_k\cos(k\omega t)+b_k\sin(k\omega t)), \tag{2.4}$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{2}\frac{\omega}{\pi}\int_0^{2\pi/\omega}(\rho(x,\dot{x},\ddot{x},t))\,\mathrm{d}t = 0,\\ a_k &= \frac{1}{2}\frac{\omega}{\pi}\int_0^{2\pi/\omega}(\rho(x,\dot{x},\ddot{x},t))\cos(k\omega t)\,\mathrm{d}t = 0,\\ b_k &= \frac{1}{2}\frac{\omega}{\pi}\int_0^{2\pi/\omega}(\rho(x,\dot{x},\ddot{x},t))\sin(k\omega t)\,\mathrm{d}t = 0,\end{aligned}\right\} \tag{2.5}$$

$k=1,\ldots,N$.

The $2N$ coefficients and frequency, ω, which characterize the periodic orbit are therefore given by $2N+1$ algebraic equations

$$\hat{G}_i(Z_L, A_{(1)L}, \omega, A_{(k)L}, B_{(k)L}: k=[2,N]) = 0, \tag{2.6}$$

where $i=[1,2N+1]$ and provided the basic assumption of the method holds we would expect the accuracy of a solution to increase with N. Essentially the above process defines a formal basis for what is in effect a method of harmonic balance for N harmonics.

2. An implicit parameter, ϵ, is now introduced for the sole purpose of devising a stability criterion. In fact ϵ is used to relate timescales s and τ to t.

$$\tau = \epsilon t, \quad \text{and} \quad s = t, \tag{2.7}$$

where s is the timescale associated with the phase variation and τ is associated with the amplitude variation. More precisely τ is the timescale related to the motion in the Poincaré section and for one-dimensional self-excited oscillators this section is one dimensional. For this class of oscillator equations, there is always a domain close to the limit cycle, the extent of which is dependent on parameters, where amplitude variations are small, and hence (for one degree of freedom autonomous equations) ϵ is small.

By means of the chain rule;

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial x}{\partial s}+\epsilon\frac{\partial x}{\partial \tau}; \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\partial^2 x}{\partial s^2}+2\epsilon\frac{\partial^2 x}{\partial s\,\partial\tau}+\epsilon^2\frac{\partial^2 x}{\partial\tau^2} \tag{2.8}$$

and equation (2.2) is transformed into

$$\frac{\partial^2 x}{\partial s^2}+\omega^2 x = (\omega^2-1)\,x-2\epsilon\frac{\partial^2 x}{\partial s\,\partial\tau}+f\left(x,\frac{\partial x}{\partial s}+\epsilon\frac{\partial x}{\partial\tau}\right)+O(\epsilon^2).$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Writing $x' = \partial x/\partial s$ and expanding f as a Taylor series, the equation above becomes

$$x'' + \omega^2 x = (\omega^2 - 1)\,x + f(x, x') - 2\epsilon\,\partial x'/\partial\tau + \epsilon\beta(x, x')\,\partial x/\partial\tau + O(\epsilon^2), \tag{2.9}$$

where terms of order ϵ^n, $n \geqslant 2$, shall be ignored in the light of $\epsilon \ll 1$ and where $\beta(x, x') = (\partial f/\partial x')$ is a known function which is independent of ϵ.

The idea is to solve equation (2.9) for trajectories close to a periodic orbit which is $2\pi/\omega$ periodic in s, and therefore a solution of similar form to (2.3) is sought with slowly varying coefficients. $A_{(k)\,\mathrm{L}}$ is replaced by $A_k(\tau)$ and $B_{(k)\,\mathrm{L}}$ replaced by $B_k(\tau)$, therefore $x(t)$ is given by

$$x(t) = Z + \sum_{k=1}^{N} (A_k(\tau)\cos(k\omega s) + B_k(\tau)\sin(k\omega s)) \tag{2.10}$$

and an equation is derived for the frequency as before by choosing the phase of the first harmonic such that $B_1(\tau) = 0$ for all time. Expression (2.10) with $B_1(\tau) = 0$ is now substituted into equation (2.9), which is then expressed as a Fourier series and the Fourier coefficients are equated to zero;

$$\begin{aligned} x'' + \omega^2 x - (\omega^2 - 1)\,x - f(x, x') + 2\epsilon\,\partial x'/\partial\tau - \epsilon\beta(x, x')\,\partial x/\partial\tau + O(\epsilon^2) \\ = \bar{\rho}(x, x', x'', \partial x/\partial\tau, \partial x'/\partial\tau, s) \\ = \tfrac{1}{2}\bar{a}_0 + \sum_{k=1}^{N} (\bar{a}_k\cos(k\omega s) + \bar{b}_k\sin(k\omega s)), \end{aligned} \tag{2.11}$$

where

$$\left.\begin{aligned} \bar{a}_0 &= \frac{1}{2}\frac{\omega}{\pi}\int_0^{2\pi/\omega}\left(\bar{\rho}\left(x, x', x'', \frac{\partial x}{\partial\tau}, \frac{\partial x'}{\partial\tau}, s\right)\right)\mathrm{d}s = 0, \\ \bar{a}_k &= \frac{1}{2}\frac{\omega}{\pi}\int_0^{2\pi/\omega}\left(\bar{\rho}\left(x, x', x'', \frac{\partial x}{\partial\tau}, \frac{\partial x'}{\partial\tau}, s\right)\right)\cos(k\omega s)\,\mathrm{d}s = 0, \\ \bar{b}_k &= \frac{1}{2}\frac{\omega}{\pi}\int_0^{2\pi/\omega}\left(\bar{\rho}\left(x, x', x'', \frac{\partial x}{\partial\tau}, \frac{\partial x'}{\partial\tau}, s\right)\right)\sin(k\omega s)\,\mathrm{d}s = 0, \end{aligned}\right\} \tag{2.12}$$

$k = 1, \ldots, N$.

Equation (2.12) represents $2N+1$ equations which can be written in the form of (2.6)

$$\left.\begin{aligned} &\hat{G}_1(Z, A_1, \omega, A_k, B_k) + D_1(Z, A_1, \mathrm{d}A_1/\mathrm{d}\tau, \omega, A_k, \quad \mathrm{d}A_k/\mathrm{d}\tau, B_k, \mathrm{d}B_k/\mathrm{d}\tau) = 0, \\ &\hat{G}_2(Z, A_1, \omega, A_k, B_k) + D_2(Z, A_1, \mathrm{d}A_1/\mathrm{d}\tau, \omega, A_k, \quad \mathrm{d}A_k/\mathrm{d}\tau, B_k, \mathrm{d}B_k/\mathrm{d}\tau) = 0, \\ &\hat{G}_i + \epsilon\left(M_i^1\frac{\mathrm{d}A_1}{\mathrm{d}\tau} + \sum_{k=2}^{N}\left(M_i^k\frac{\mathrm{d}A_k}{\mathrm{d}\tau} + N_i^k\frac{\mathrm{d}B_k}{\mathrm{d}\tau}\right)\right) = 0, \quad i = [3, 2N+1], k = [2, N], \end{aligned}\right\} \tag{2.13}$$

where $\hat{G}_i, M_i^1, M_i^k, N_i^k$ are all functions of $Z, A_1(\tau), \omega, A_k(\tau), B_k(\tau) : k = [2, N]$, $i = [3, 2N+1]$, and on the limit cycle $D_1 = D_2 = 0$.

Clearly the difference between (2.6) and (2.13) is of order ϵ, and therefore the differences between $A_1(\tau)$ and $A_{(1)\,\mathrm{L}}$, $A_k(\tau)$ and $A_{(k)\,\mathrm{L}}$, and $B_k(\tau)$ and $B_{(k)\,\mathrm{L}}$, $k = [2, N]$ must also be of order ϵ. For notational convenience $(A_{(1)\,\mathrm{L}}, A_{(k)\,\mathrm{L}}, B_{(k)\,\mathrm{L}} : k = [2, N])$ will be written as the vector $\boldsymbol{H}_\mathrm{L}$, and $(A_1(\tau), A_k(\tau), B_k(\tau) : k = [2, N])$ as $\boldsymbol{H}(\tau)$. Therefore, if the vector $\boldsymbol{M}_i(Z,\omega,H(\tau))$ represents $(M_i^1, M_i^k, N_i^k : k=[2,N])$, $i=[3,2N+1]$, a Taylor expansion about $\boldsymbol{H}_{\mathrm{L}}$ gives

$$\hat{G}_i(Z_{\mathrm{L}},\omega_{\mathrm{L}},\boldsymbol{H}_{\mathrm{L}})+(\partial\hat{G}_i/\partial\boldsymbol{H})\cdot(\boldsymbol{H}(\tau)-\boldsymbol{H}_{\mathrm{L}}) = \epsilon\boldsymbol{M}_i(Z_{\mathrm{L}},\omega_{\mathrm{L}},\boldsymbol{H}_{\mathrm{L}})\cdot(\mathrm{d}/\mathrm{d}\tau)(\boldsymbol{H}(\tau)-\boldsymbol{H}_{\mathrm{L}}) + \text{higher-order terms}, \quad i=[3,2N+1],$$

where for two vectors $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^{2N-1}$,

$$\boldsymbol{a}\cdot\boldsymbol{b} = \sum_{j=1}^{2N-1} a_j b_j.$$

By equation (2.6) and writing $\boldsymbol{H}(\tau)-\boldsymbol{H}_{\mathrm{L}}$ as $\boldsymbol{z}$ the above equation becomes retaining only linear terms

$$\epsilon\boldsymbol{M}_i(Z_{\mathrm{L}},\omega_{\mathrm{L}},\boldsymbol{H}_{\mathrm{L}})\cdot\mathrm{d}\boldsymbol{z}/\mathrm{d}\tau = (\partial\hat{G}_i/\partial\boldsymbol{H})\cdot\boldsymbol{z}, \quad i=[3,2N+1], \tag{2.14}$$

which will determine the stability of the periodic orbit given by (2.10).

If the matrix M has the vectors $\boldsymbol{M}_i$, $i=[3,2N+1]$, as its $2N-1$ columns and the vector $\boldsymbol{G}$ represents $(\hat{G}_i : i=[3,2N+1]))$ then expression (2.13) can be written in a compact form:

$$\boldsymbol{G}(Z,\omega,\boldsymbol{H})+\epsilon(\mathrm{d}\boldsymbol{H}/\mathrm{d}\tau)M(Z,\omega,\boldsymbol{H})=0,$$
$$Z=\hat{Z}(\omega,\boldsymbol{H},\epsilon\,\mathrm{d}\boldsymbol{H}/\mathrm{d}\tau)=Z(\omega,\boldsymbol{H}),$$
$$\omega=\hat{\omega}(Z,\boldsymbol{H},\epsilon\,\mathrm{d}\boldsymbol{H}/\mathrm{d}\tau)=\omega(Z,\boldsymbol{H}).$$

A trajectory close to an asymmetric limit cycle solution has an amplitude that varies with τ and as a consequence the mean of oscillation, Z, and frequency, ω, are affected since they depend on the amplitude, $\boldsymbol{H}(\tau)$. Note that Z does not vary independently since this would not describe a neighbouring trajectory in these one-dimensional autonomous oscillators.

3. Applications

In this section 2THB will be applied to three autonomous equations: the van der Pol equation (van der Pol 1922),

$$\ddot{x}+(x^2-\mu)\,\dot{x}+x=0; \tag{3.1}$$

the modified van der Pol equation;

$$\ddot{x}+(\beta x^4+x^2-\mu)\,\dot{x}+x=0; \tag{3.2}$$

the van der Pol equation with escape,

$$\ddot{x}+(x^2-\mu)\,\dot{x}+x-\gamma x^2=0, \tag{3.3}$$

each of which could be recast by a suitable transformation to make μ a measure of the nonlinear terms. For μ small equation (3.3) could be analysed by Kuzmak's method; however, 2THB imposes no restriction on μ provided the basic assumption is valid, and results are compared with numerical solutions. In each case we shall seek solutions both when μ is small and order unity and see how the number of harmonics, N, required for an accurate representation for $x(t)$ depends directly on the values of μ, β, and γ. Whenever $\mu \ll 1$, which from the linearized equation is seen to correspond to slow decay over a large domain of the one-dimensional Poincaré section, solutions can be obtained that are identical with the method of multiple scales to order μ, see

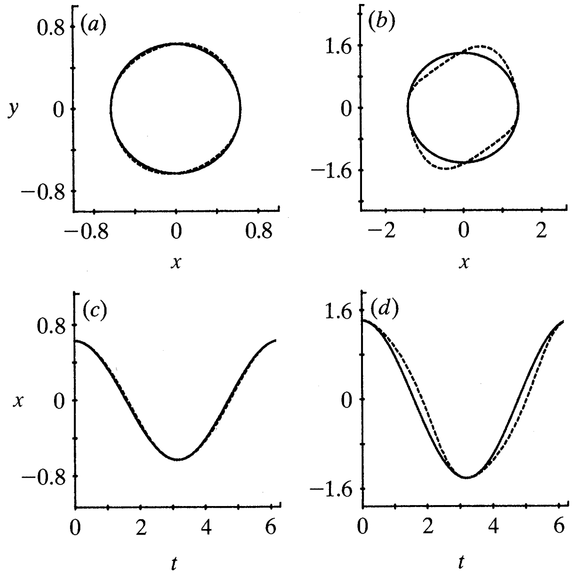

Figure 1. Comparison of a one harmonic approximation with numerical results for the small μ limit cycle solution of the van der Pol equation (3.1). ——, 2THB; ----, numerical. (*a*), (*c*) $\mu = 0.1$; (*b*), (*d*) $\mu = 0.5$.

Appendix A. However, when μ is $O(1)$ the domain for which τ is slow compared with s (implying ϵ small) is much smaller.

(*a*) *van der Pol equation*

With two timescales introduced equation (3.1) reduces to a form analogous to equation (2.9)

$$x'' + \omega^2 x = (\omega^2 - 1)\,x - (x^2 - \mu)\,x' - 2\epsilon\,\partial x'/\partial\tau - \epsilon(x^2 - \mu)\,\partial x/\partial\tau + O(\epsilon^2). \qquad (3.4)$$

(i) $\mu \ll 1$

A first harmonic approximation to $x(t)$ is given by

$$x(t) = A(\tau)\cos(\omega s) + B(\tau)\sin(\omega s)$$

in which, without loss of generality, $B(\tau)$ can be set to zero and there is no need to include a displacement Z due to a result obtained in Appendix B,

$$x(t) = A(\tau)\cos(\omega s). \qquad (3.5)$$

Substituting for $x(t)$ via (3.5) into equation (3.4) and expressing the result as a Fourier series (cf. equation (2.11)) gives

$$\bar{a}_1 = 0 \Rightarrow \epsilon(\mathrm{d}A/\mathrm{d}\tau)\,(\tfrac{3}{4}A^2 - \mu) = (\omega^2 - 1)\,A, \qquad (3.6a)$$

$$\bar{b}_1 = 0 \Rightarrow \epsilon\,\mathrm{d}A/\mathrm{d}\tau = \tfrac{1}{2}A(\mu - \tfrac{1}{4}A^2). \qquad (3.6b)$$

Interpretation. 1. Stationary solutions of (3.6) corresponding to limit cycle solutions for $x(t)$ are given by

$$A = A_\mathrm{L} = 2\sqrt{\mu} \quad \text{and} \quad \omega = \omega_\mathrm{L} = 1.$$

In figure 1 comparison is made between the analytical and numerical solutions for $y = \mathrm{d}x/\mathrm{d}t$ against x and x against t, for two values of μ; $\mu = 0.1$, there is very close agreement over the whole cycle; $\mu = 0.5$, there is clearly a difference between the two solutions illustrating the need for higher harmonics in the solution for $x(t)$.

2. A simple stability argument follows from the evolution equation, (3.6*b*). The solution with $A = 2\sqrt{\mu}$ is stable since: when $A < 2\sqrt{\mu}$; $\mathrm{d}A/\mathrm{d}\tau > 0$ and therefore A will increase to $2\sqrt{\mu}$ as t increases, similarly when $A > 2\sqrt{\mu}$; $\mathrm{d}A/\mathrm{d}\tau < 0$ and A will decrease to $2\sqrt{\mu}$ as t increases.

3. From the above solution, A is $O(\mu^{\frac{1}{2}})$ and therefore (3.6*b*) implies that ϵ is $O(\mu)$ which means that amplitude variations occur on a slow timescale of order (μt).

4. Equation (3.6*a*) can be rewritten via (3.6*b*) in the form

$$\omega^2 - 1 = \tfrac{1}{2}(\mu - \tfrac{1}{4}A^2)(\tfrac{3}{4}A^2 - \mu),$$

where the right-hand side is $O(\mu^2)$ and can therefore be neglected. The solution to $O(\mu)$ is therefore

$$\omega = 1; \quad \epsilon\,\mathrm{d}A/\mathrm{d}\tau = \tfrac{1}{2}A(\mu - \tfrac{1}{4}A^2), \tag{3.6c}$$

which is similar to van der Pol's own approximation. This solution is identical to that obtained by multiple scales to order μ, which then raises the question of whether this is always true for any autonomous second-order equations with a small parameter. It is shown in Appendix A, that solutions obtained by 2THB and the method of multiple scales, to order μ, are generally the same provided the solution includes first and second harmonics. In the case of the van der Pol equation, however, the second harmonic does not contribute to the solution, as a consequence of a result established in Appendix B for ODEs with antisymmetric vector fields. It is shown that only odd harmonics appear in the solution $x(t)$ of any second-order autonomous dynamical system which is antisymmetric, i.e. possesses an antisymmetric vector field.

5. If μ is negative but remains small, the solution of equation (3.6) is $A_{\mathrm{L}} = 0$ and ω is undetermined, implying that there are no periodic solutions for μ negative but there is a stationary solution at the origin. When μ is positive, $A_{\mathrm{L}} = 2\sqrt{\mu}$ and $\omega_{\mathrm{L}} = 1$ is a solution as discussed above, but $A_{\mathrm{L}} = 0$ and ω arbitrary is still a solution of (3.6). Therefore as μ goes through zero from negative to positive, the number of solutions of (3.6) changes from one to two. A change in the character of the solution takes place at $\mu = 0$, which is a feature of the Hopf bifurcation at $(x, \mu) = (0, 0)$. By a similar argument to that of (2) above, the stability of the origin can be determined for μ negative (stable) and positive (unstable).

(ii) $\mu = O(1)$

1. *The solution.* When the control parameter μ is no longer small a first harmonic solution for $x(t)$ is generally insufficient (cf. figure 1) and there is need for an N harmonic expansion (cf. equation (2.10)), and as before the phase of the first harmonic is chosen to be zero (i.e. $B_1 = 0$).

The procedure is as before, where the expansion given by (2.10) is substituted into (2.11) a set of $2N+1$ equations are produced from equations (2.12), two of which are algebraic, and $2N-1$ involve first derivatives of $\boldsymbol{H}$, where $\boldsymbol{H}$ is the vector of the $2N-1$ coefficients $(A_1, A_k, B_k : k = [2, N])$ of the Fourier expansion (2.10)

$$\bar{a}_0 = 0 \Rightarrow \Xi(Z, \boldsymbol{H}, \omega, \boldsymbol{\lambda}) + \epsilon(\mathrm{d}\boldsymbol{H}/\mathrm{d}\tau)\cdot \boldsymbol{C}_Z = 0, \tag{3.7a}$$

$$\bar{a}_1 = 0 \Rightarrow \Omega(Z, \boldsymbol{H}, \omega, \boldsymbol{\lambda}) + \epsilon(\mathrm{d}\boldsymbol{H}/\mathrm{d}\tau)\cdot \boldsymbol{C}_\omega = 0, \tag{3.7b}$$

$$\bar{b}_1 = \bar{a}_k = \bar{b}_k = 0 \Rightarrow \epsilon(\mathrm{d}\boldsymbol{H}/\mathrm{d}\tau) M(Z, \boldsymbol{H}, \omega, \boldsymbol{\lambda}) + \boldsymbol{G}(Z, \boldsymbol{H}, \omega, \boldsymbol{\lambda}) = \boldsymbol{0},$$
$$k = [2, N]\, \boldsymbol{H}, \boldsymbol{G} \in \mathbb{R}^{2N-1}, \tag{3.7c}$$

where $\boldsymbol{G}(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda})$ is the vector of terms from the integrals, $\bar{b}_1, \bar{a}_k, \bar{b}_k$, $k=[2,N]$, in (2.12) which do not involve τ derivatives, i.e. equations (2.5). The matrix $M(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda})$ is $2N-1\times 2N-1$, $\Xi(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda})=0$ and $\Omega(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda})=0$ are algebraic equations which give the mean and the frequency of the limit cycle solution ($H=H_{\mathrm{L}}$), that arise from balancing the non-oscillatory terms and $\cos(\omega s)$ coefficients respectively. $\boldsymbol{\lambda}$ is the vector of control parameters such that $\boldsymbol{\lambda}=(\mu)$, $\boldsymbol{\lambda}=(\mu,\beta)$, and $\boldsymbol{\lambda}=(\mu,\gamma)$ in equations (3.1), (3.2), and (3.3) respectively. Equation (3.7*b*) enables the frequency of the periodic solution to be evaluated, and is dependent on $\boldsymbol{H}$ (which in turn depends on $\boldsymbol{\lambda}$), so that it can be written explicitly as $\omega=\omega(Z,\boldsymbol{H},\boldsymbol{\lambda})$. By inverting matrix M, (3.7*c*) becomes a set of autonomous equations which describe the time evolution of the Fourier coefficients.

$$\epsilon\,\mathrm{d}\boldsymbol{H}/\mathrm{d}\tau=-\boldsymbol{G}(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda})\,(M(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda}))^{-1}=\boldsymbol{F}(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda}) \tag{3.8a}$$

and $\mathrm{d}\boldsymbol{H}/\mathrm{d}\tau$ is $O(1)$.

$$\Xi(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda})+\boldsymbol{F}(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda})\cdot\boldsymbol{C}_Z=0\Rightarrow Z=Z(\boldsymbol{H},\omega,\boldsymbol{\lambda}), \tag{3.8b}$$

$$\Omega(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda})+\boldsymbol{F}(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda})\cdot\boldsymbol{C}_\omega=0\Rightarrow \omega=\omega(Z,\boldsymbol{H},\boldsymbol{\lambda}). \tag{3.8c}$$

The problem of seeking periodic solutions of equation (2.2) is replaced by that of finding stationary values for the coefficients of the expansion (2.10), where $\boldsymbol{F}\in\mathbb{R}^{2N-1}$. Note that $M(Z,\boldsymbol{H},\omega,\boldsymbol{\lambda})$ is dependent on the term $\epsilon\beta(x,x')\,\partial x/\partial\tau$, which in turn is dependent on $\boldsymbol{\lambda}$.

The stationary solutions of (3.8) are given by $\mathrm{d}\boldsymbol{H}/\mathrm{d}\tau=0$, i.e. $\boldsymbol{H}=\boldsymbol{H}_{\mathrm{L}}$ where $F(Z_{\mathrm{L}},\boldsymbol{H}_{\mathrm{L}},\omega_{\mathrm{L}},\boldsymbol{\lambda})=0$. These stationary values of the coefficients yield the solution written explicitly by using (2.10) as

$$x_{\mathrm{L}}(t)=Z_{\mathrm{L}}+A_{(1)\,\mathrm{L}}\cos(\omega_{\mathrm{L}}s)+\sum_{k=2}^{N}(A_{(k)\,\mathrm{L}}\cos(k\omega_{\mathrm{L}}s)+B_{(k)\,\mathrm{L}}\sin(k\omega_{\mathrm{L}}s))$$

and

$$\frac{\mathrm{d}x_L}{\mathrm{d}t}=y_{\mathrm{L}}=-\omega_{\mathrm{L}}A_{(1)\,\mathrm{L}}\sin(\omega_{\mathrm{L}}s)+\sum_{k=2}^{N}k\omega_{\mathrm{L}}(-A_{(k)\,\mathrm{L}}\sin(k\omega_{\mathrm{L}}s)+B_{(k)\,\mathrm{L}}\cos(k\omega_{\mathrm{L}}s)),$$

where
$$\boldsymbol{H}_{\mathrm{L}}=(A_{(1)\,\mathrm{L}}(\boldsymbol{\lambda}),A_{(k)\,\mathrm{L}}(\boldsymbol{\lambda}),B_{(k)\,\mathrm{L}}(\boldsymbol{\lambda}):k=[2,N]),$$
$$\omega_{\mathrm{L}}=\omega(Z_{\mathrm{L}},\boldsymbol{H}_{\mathrm{L}},\boldsymbol{\lambda})\quad\text{and}\quad Z_{\mathrm{L}}=Z(\boldsymbol{H}_{\mathrm{L}},\omega_{\mathrm{L}},\boldsymbol{\lambda}).$$

To actually obtain the equations represented by (3.8) is algebraically tedious and susceptible to human error for more than one harmonic, and it is therefore convenient to make use of a symbolic manipulator. Equally the analysis of the equations is not a straightforward task, and it is helpful to have use of a path following package with the facility to follow stationary solutions of a $(2N-1)$-order dynamical system with up to two control parameters. Throughout this work the authors have used the computer algebra package, REDUCE (cf. manual by Rayna 1987) and the path following routine, PATH (Kaas-Petersen 1987).

Returning to the application of 2THB to the van der Pol equation (3.1), the resulting equations are evaluated with harmonic expansions of two, four, and five harmonics. A set of these equations for five harmonics is given in Summers (1991). Figures 2 and 3 respectively demonstrate the results of a three and seven harmonic

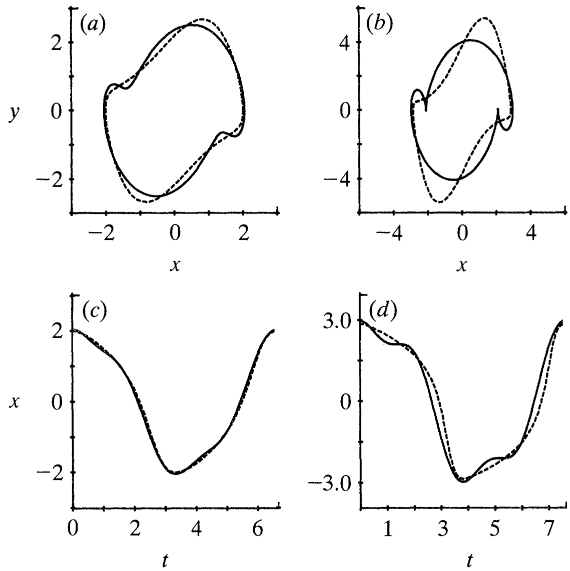

Figure 2. Comparison of a two harmonic approximation with numerical results of the oscillation for (a), (c) $\mu = 1.00$ and (b), (d) $\mu = 2.00$ in equation (3.1). ——, 2THB; ----, numerical.

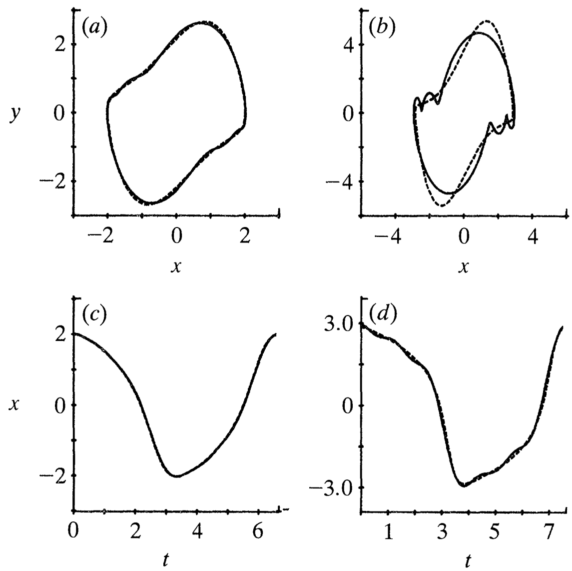

Figure 3. Comparison of a four harmonic approximation with numerical results of the oscillation for (a), (c) $\mu = 1.00$ and (b), (d) $\mu = 2.00$ in equation (3.1). ——, 2THB; ----, numerical.

expansion, each with two values of μ, ($\mu = 1.0$ and $\mu = 2.0$). On the left side of figure 4 results are shown for a nine harmonic expansion with $\mu = 2.0$ and on the right x–t plots are shown for $\mu = 3.0$ and 5.0 for a 51 harmonic expansion. In each case it can be seen that the analytical solution given by the expansion tends to the numerical solution as the number of harmonics is increased. Also for $\mu = 3.0$ and 5.0 there are two parts of the limit cycle where the gradient changes sharply indicating the feature of a relaxation oscillation which is well established when μ is large. For $\mu = 5.0$ the

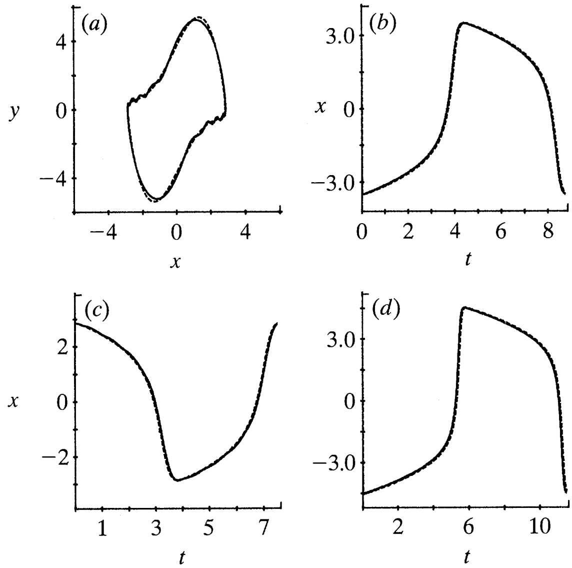

Figure 4. Comparison of a five harmonic approximation with numerical results of the oscillation for (*a*), (*c*) $\mu = 2.00$ and comparisons of the oscillations for a 51 harmonic expansion with numerical results when (*b*) $\mu = 3.00$ and (*d*) $\mu = 5.00$ in equation (3.1). ——, 2THB; ----, numerical.

51 harmonic expansion was needed to achieve close agreement between analytic and numerical curves of $x(t)$ against t. The right-hand side of figure 4 illustrates that 2THB can, in principle, yield solutions for 'weak' relaxation oscillations yet in practice the number of harmonics required soon becomes excessively large. This illustrates that there is a restriction on μ according to the number of terms that can be handled by the symbolic manipulator.

Figures 3 and 4 also reveal that agreement between analytic and numerical results is best illustrated by plotting $x(t)$ against t rather than limit cycle orbits which exhibit small oscillations (the amplitude of which diminishes with increasing N). This is to be expected since a key feature of a relaxation oscillation is a rapid change in $\dot{x}$ (and even greater change in $\ddot{x}$) which will require more harmonics than in the expansion for x.

A clear justification for writing the solution, $x(t)$, as a Fourier series is observed by analysing the relative frequency spectra, where it can be seen how the energy of the oscillation is distributed among the higher harmonics. In figure 5 a comparison between the numerical and analytical relative frequency spectra is made for $\mu = 3.0$.

2. *The stability*. To construct a stability argument for a many harmonic expansion, the stability of the Fourier coefficients must be considered. Clearly the solution $x(t)$ will be stable if all of these coefficients, whose evolution is given by equation (3.8), are stable. The usual procedure for determining the stability of stationary solutions of the dynamical system (3.8) is to linearize about the solution $\boldsymbol{H}_L$. Setting $\boldsymbol{H} = \boldsymbol{H}_L + z$ with $|z| \ll 1$, substituting into equation (3.8) and expanding $\boldsymbol{F}$ as a Taylor series gives (cf. (2.14))

$$\epsilon\, \mathrm{d}z/\mathrm{d}\tau = \boldsymbol{F}(Z_L, \boldsymbol{H}_L, \omega_L, \boldsymbol{\lambda}) + L(Z_L, \boldsymbol{H}_L, \omega_L, \boldsymbol{\lambda})\, z = L(Z_L, \boldsymbol{H}_L, \omega_L, \boldsymbol{\lambda})\, z, \tag{3.9}$$

where the matrix L is the jacobian of $\boldsymbol{F}$ at $\boldsymbol{H} = \boldsymbol{H}_L$, $\omega_L = \omega(Z_L, \boldsymbol{H}_L, \boldsymbol{\lambda})$, and $Z_L = Z(\boldsymbol{H}_L, \omega_L, \boldsymbol{\lambda})$. The eigenvalues of the matrix L can then be calculated to determine the

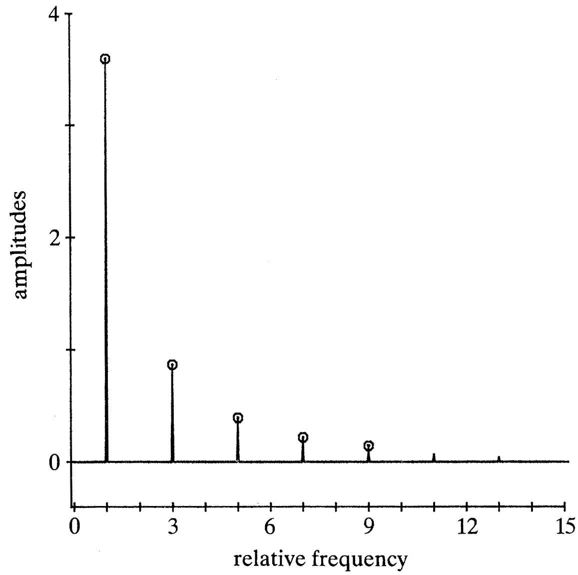

Figure 5. Numerical and analytical relative frequency spectra for $\mu = 3.00$ in equation (3.1). ——, Numerical spectra; ⊙, 2THB amplitudes.

stability of the solution, $\boldsymbol{H}_L$, thus determining the stability of the Fourier coefficients, $\boldsymbol{H}$. Increased saving in computing time is achieved if the path following routine has a facility for calculating the eigenvalues of L at every step on the path.

With the presence of a control parameter in the governing equation the solution may undergo a change in stability at a particular value of this parameter, known as a bifurcation point. This can be determined by analysing the eigenvalues of L (cf. equation (3.9)) associated with the solution of expression (3.8) as the control parameter is varied. Again this is facilitated by the use of a path following package such as PATH. In the van der Pol equation with μ negative, a solution is determined to the set of equations (3.8) yielding $\boldsymbol{H}_L = \boldsymbol{0}$ with all the eigenvalues of L having a negative real part. This identifies the stationary solution and its stability at the origin of phase space. By following the path of the solution as μ is increased, the bifurcation at $\mu = 0$ is found where one eigenvalue of L passes through the origin of the complex plane, and simultaneously the remaining eigenvalues pass into the positive half of the complex plane with non-zero imaginary part. The origin has now become unstable and the amplitudes of the expansion will grow from zero with increasing time. A complete bifurcation analysis at $\mu = 0$ using PATH, shows that the eigenvalue of L passing through the origin of the complex plane indicates the existence of a bifurcating branch which has its tangent pointing in the direction of the amplitude growth of the first harmonic. The remaining eigenvalues indicate a type of equivariant bifurcation and from the viewpoint of 2THB the interpretation of this bifurcation indicates a growth in the coefficients of the higher harmonics at $\mu = 0$. When μ is positive the origin of the phase space is unstable as every eigenvalue of L has positive real part. At $(x, \mu) = (0, 0)$ there are two branches that can be followed, one corresponding to the stationary point and the other to the periodic solutions. If the bifurcating branch of periodic solutions is followed (by a path following package) the eigenvalues of L for this second solution all have negative real part and the solution branch is therefore stable as expected.

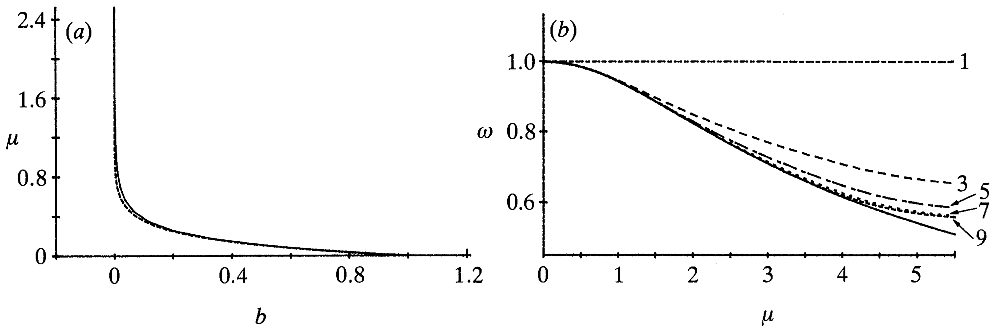

Figure 6. (*a*) Numerical (----) and analytical (——) variation with the parameter μ, of the characteristic multiplier for equation (3.1). (*b*) Frequency variation with the parameter μ, for different harmonic expansions for the solution of equation (3.1). ——, numerical.

In numerical investigations the problem of stability is analysed by using the notion of a Poincaré map, and in the case of second-order autonomous dynamical systems this map would be one dimensional. There is a need therefore to show that the method of 2THB produces a stability argument consistent with the Poincaré map approach. This is achieved by finding an estimate, via 2THB, of the characteristic multiplier of the one-dimensional map and comparing with the numerically obtained value. Figure 6 shows this comparison for the van der Pol equation and the detailed mathematical analysis for estimating the characteristic multiplier b of a one-dimensional map is given in Summers (1991). Figure 6 also shows the variation of the frequency with μ for various harmonic expansions and compares these results with numerical results.

(*b*) *Modified van der Pol equation*

Introducing two timescales equation (3.2) reduces to the form

$$x'' + \omega^2 x = (\omega^2 - 1)\,x - (\beta x^4 + x^2 - \mu)\,x' - 2\epsilon \frac{\partial x'}{\partial \tau} - \epsilon(\beta x^4 + x^2 - \mu)\frac{\partial x}{\partial \tau} + O(\epsilon^2). \quad (3.10)$$

(i) $\mu \ll 1$

A first harmonic approximation of $x(t)$ is

$$x(t) = A(\tau)\cos(\omega s), \quad (3.11)$$

which is substituted into equation (3.10) and the result expressed as a Fourier series (cf. equation (2.7)) giving

$$\bar{a}_1 = 0 \Rightarrow \epsilon\, \mathrm{d}A/\mathrm{d}\tau(\mu - \tfrac{3}{4}A^2 - \tfrac{5}{8}\beta A^4) = (1-\omega^2)A, \quad (3.12a)$$

$$\bar{b}_1 = 0 \Rightarrow \epsilon\, \mathrm{d}A/\mathrm{d}\tau = \tfrac{1}{16}A(-\beta A^4 - 2A^2 + 8\mu). \quad (3.12b)$$

Interpretation. 1. Stationary solutions have $\omega = \omega_{\mathrm{L}} = 1$ and amplitude A_{L} where

$$\beta A_{\mathrm{L}}^4 + 2A_{\mathrm{L}}^2 - 8\mu = 0,$$

$$A_{\mathrm{L}}^2 = [-1 \pm (1+8\mu\beta)^{\frac{1}{2}}]/\beta.$$

(*a*) If $\beta > 0$ there is one non-zero solution for A_{L}^2

$$A_{\mathrm{L}}^2 = \beta^{-1}(\sqrt{(1+8\mu\beta)} - 1).$$

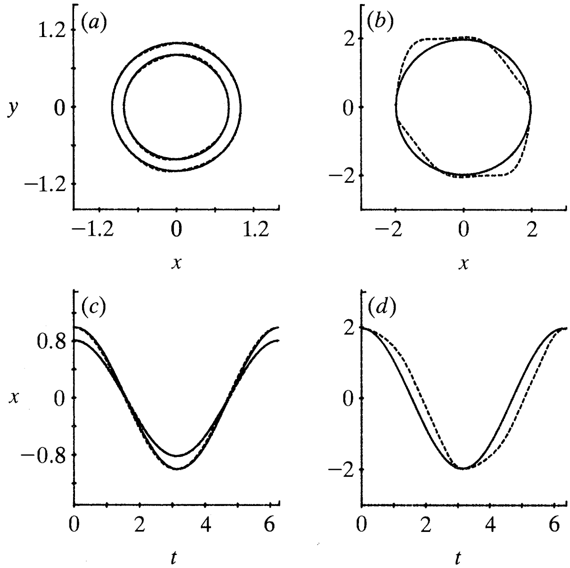

Figure 7. Comparison of a one harmonic approximation with numerical results for the modified van der Pol equation (3.2); for (*a*), (*c*) $(\mu, \beta) = (0.10, -1.20)$ both the stable and unstable orbits are shown; and for (*b*), (*d*) $(\mu, \beta) = (0.04, -0.49)$ only the unstable orbit is illustrated. ——, 2THB; ----, numerical.

(i) If $\mu > 0$ and $8\mu\beta \ll 1$, then $A_{\mathrm{L}} \approx 2\sqrt{\mu}$.

(ii) If β is large and $8\mu\beta$ is no longer small, then A_{L} is $O((\mu/\beta)^{\frac{1}{4}})$.

(*b*) If $\beta < 0$ there are two non-zero solutions for A with $\omega_{\mathrm{L}} = 1$:

$$A^2_{\mathrm{L(i)}} = -\beta^{-1}(1+\sqrt{(1+8\mu\beta)}), \quad \mu < -1/8\beta,$$

and

$$A^2_{\mathrm{L(ii)}} = -\beta^{-1}(1-\sqrt{(1+8\mu\beta)}), \quad 0 < \mu < -1/8\beta.$$

There are, therefore, coexistent limit cycles when $\beta < 0$ and $0 < \mu < -1/8\beta$.

(i) If $1+8\mu\beta = \delta^2$ with $\delta \ll 1$, the two limit cycles have amplitudes

$$A_{\mathrm{L(i)}} \approx (1+\tfrac{1}{2}\delta)(-1/\beta)^{\frac{1}{2}}; \quad A_{\mathrm{L(ii)}} \approx (1-\tfrac{1}{2}\delta)(-1/\beta)^{\frac{1}{2}}, \tag{3.13}$$

i.e. of $O(2\sqrt{(2\mu)})$ and so both orbits are circular since μ is small.

(ii) If $|8\mu\beta| \ll 1$ and β is $O(1)$, the two limit cycles have amplitudes

$$A_{\mathrm{L(i)}} \approx (-2/\beta)^{\frac{1}{2}} \quad \text{and} \quad A_{\mathrm{L(ii)}} \approx 2\sqrt{\mu}$$

corresponding to large (non-circular) and small (circular) orbits respectively.

Figure 7 shows these coexistent limit cycles obtained numerically and by (3.12) for case (i) $(\mu, \beta) = (0.1, -1.2)$ and for case (ii) (only the unstable periodic orbit is shown) $(\mu, \beta) = (0.04, -0.49)$. For the former case, as expected close agreement is found when $1+8\mu\beta$ is small and β is $O(1)$ whereas the latter case shows the need to include higher harmonics in the solution for the unstable limit cycle.

2. As before a stability argument follows from the amplitude evolution equation (3.12*b*). With β negative and $0 < A < A_{\mathrm{L(ii)}} < A_{\mathrm{L(i)}}$ then $\mathrm{d}A/\mathrm{d}\tau > 0$ and A will increase to $A_{\mathrm{L(ii)}}$ as t increases. For $A_{\mathrm{L(ii)}} < A < A_{\mathrm{L(i)}}$, $\mathrm{d}A/\mathrm{d}\tau < 0$ and A will decrease to $A_{\mathrm{L(ii)}}$ as t increases. Therefore the solution $A = A_{\mathrm{L(ii)}}$ is stable.

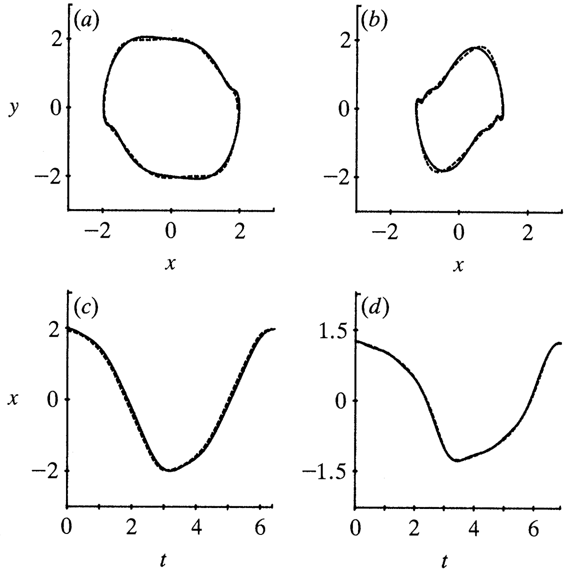

Figure 8. Comparison of a three harmonic approximation with numerical results of (*a*), (*c*) an unstable orbit ($(\mu, \beta) = (0.04, -0.49)$) and (*b*), (*d*) a stable orbit ($(\mu, \beta) = (1.0, 2.0)$) for the solution of equation (3.2). ——, 2THB; ----, numerical.

Similarly when $A_{L(ii)} < A_{L(i)} < A$; $dA/d\tau > 0$, A will increase and move away from $A_{L(i)}$ as t increases. Therefore the solution $A = A_{L(i)}$ is unstable.

3. When A_L is $O(\mu^{\frac{1}{2}})$ then ϵ is $O(\mu)$ via equation (3.12*b*) and so (μt) is a characteristic slow timescale for amplitude variations. Furthermore the term on the left-hand side of this equation is $O(\mu^2)$ and the solution to $O(\mu)$ is $\omega_L = 1$. Identical solutions would be obtained by multiple scales, to order μ for μ small, as a consequence of the results of Appendixes A and B.

4. When μ is negative and β positive the only solution is $A_L = 0$ and ω arbitrary, and when β and μ are both negative the zero solution still exists with the same stability, but there is a limit cycle given by $A_L = A_{L(i)}$. It is clear that for all values of β there is a Hopf bifurcation at $\mu = 0$, and when β is negative there is a curve in control space, (μ, β), given by $\mu\beta = -\frac{1}{8}$, where to the left there are two non-zero amplitude solutions while to the right there are no solutions with non-zero amplitudes. The point in (μ, β) space where $\mu\beta = -\frac{1}{8}$ is a first approximation of the saddle node bifurcation (or fold bifurcation) that occurs to the solution of the modified van der Pol equation, and can be seen from equation (3.13) as the coalescence of the two solutions as $\delta \to 0$. See figure 11 for the bifurcation curve in control space.

(ii) $\mu = O(1)$

1. *The solution.* The formalization of the approach for an N harmonic expansion is given in §3*a*. The application of 2THB to the modified van der Pol equation is similarly performed with two and three harmonics, and the results for the expansion of five harmonics are shown in figure 8, with two values of μ and β, ($\mu = 0.04$, $\beta = -0.49$, and $\mu = 1.0$, $\beta = 2.0$). The agreement between analytical and numerical results for the unstable limit cycle at $(\mu, \beta) = (0.04, -0.49)$ can be viewed alternatively by comparing their relative frequency spectra shown in figure 9.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

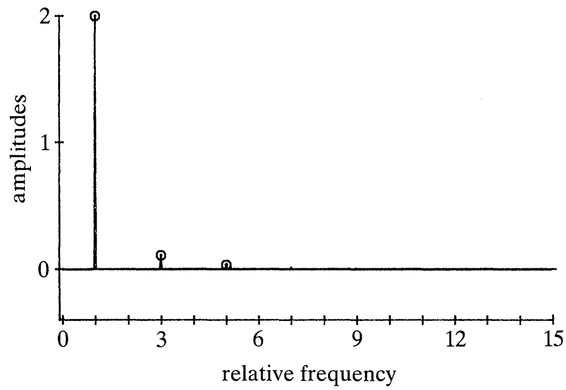

Figure 9. Numerical and analytical relative frequency spectra for $(\mu, \beta) = (0.04, -0.49)$ in equation (3.2). ——, Numerical spectra; ⊙, 2THB amplitudes.

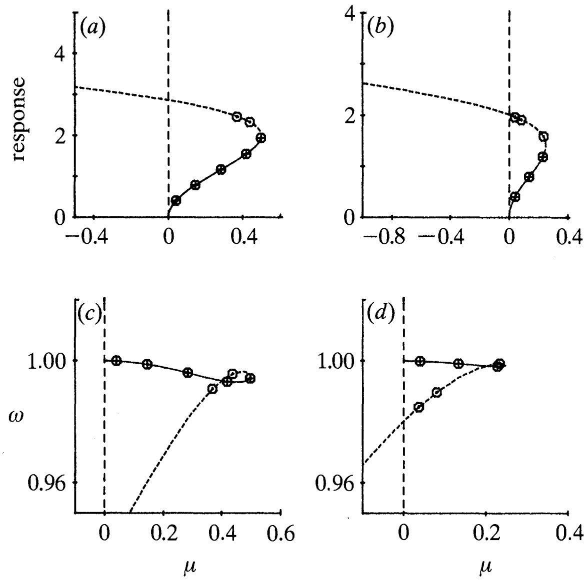

Figure 10. (*a*), (*c*) Numerical and analytical parameter-response curves for equation (3.2) $(\beta = -0.25)$. (*b*), (*d*) Numerical and analytical frequency variation with μ for equation (3.2) $(\beta = -0.50)$. 2THB: ----, unstable; ——, stable. Numerical: ⊙, unstable; ⊕, stable.

Parameter-response diagrams can be obtained with very little computational effort, and since there is a second control parameter, β, present in the modified van der Pol equation, the maximum response versus the first control parameter, μ, can be produced for various values of β. Figure 10 shows two such diagrams for $\beta = -0.25$ and -0.50 along with the μ variation of the frequency of the periodic solution and the numerical solutions are superimposed. However, it can be shown that the numerical solutions stop prematurely, this is because the continuation routines have difficulty proceeding along the path of unstable orbits as the orbits are no longer smooth but squarelike. 2THB correctly predicts the existence of the unstable limit cycle for decreasing μ, but requires more harmonics to accurately represent it.

2. *The stability*. The eigenvalues of the jacobian matrix for more than one

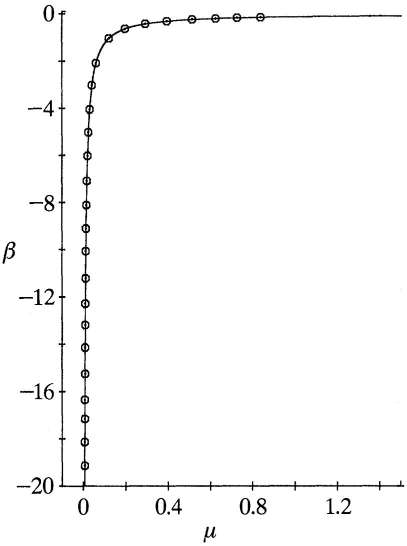

Figure 11. Control space (μ, β) diagram depicting the boundary of the limit cycle solutions, and the curve indicates the fold bifurcation. ——, 2THB; ⊙, numerical.

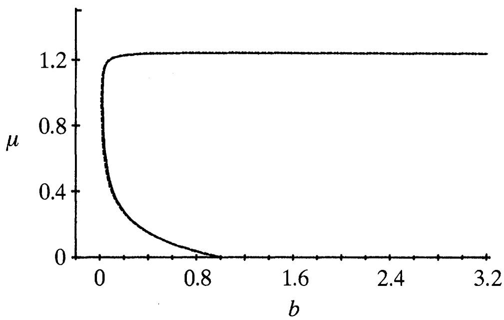

Figure 12. Numerical and analytical variation with the parameter μ, of the characteristic multiplier for $\beta = -0.10$ in equation (3.2)

harmonic are evaluated from equation (3.9) in the same way as discussed in §3*a*. The eigenvalues of matrix L of (3.9) determine the stability of the solution (be it periodic or stationary). The modified van der Pol equation undergoes a Hopf bifurcation at $\mu = 0$ for any value of β, however the magnitude of the ensuing periodic orbit is β dependent. The complete bifurcation analysis for a Hopf bifurcation, when applying 2THB, was discussed in §3*a*, but the modified van der Pol equation exhibits a Fold bifurcation which is identified when one eigenvalue crosses through the origin from negative to positive. Analogous to the Hopf bifurcation at $\mu = 0$, eigenvalues associated with the higher harmonics cross the imaginary axis in conjugate pairs indicating that all the coefficients change stability at the fold. In figure 11 a control space (μ, β) diagram shows the curve of folds from the fifth-order system and compares this with the numerically obtained curve.

Finally the characteristic multiplier first introduced in §3*a* is evaluated for the modified van der Pol equation, and compared with its numerically obtained value. Figure 12 shows such a comparison for a three harmonic expansion at $\beta = -0.1$, and when β is negative there is a saddle node bifurcation at a particular value of μ, which

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

is observed in figure 12 where the curve passes through $b = 1$, so that $b > 1$ indicates that the periodic orbit is unstable.

(c) *van der Pol equation with escape*

With the inclusion of the two timescales, equation (3.3) becomes

$$x'' + \omega^2 x = (\omega^2 - 1)\,x - (x^2 - \mu)\,x' + \gamma x^2 - 2\epsilon\,\partial x/\partial\tau - \epsilon(x^2-\mu)\,\partial x/\partial\tau + O(\epsilon^2). \quad (3.14)$$

(i) $\mu \ll 1$

An equation without an antisymmetric vector field requires a two harmonic approximation, including a mean of oscillation, for an agreement with multiple scales to $O(\mu)$ (cf. Appendixes A and B):

$$x(t) = Z + A(\tau)\cos(\omega s) + C(\tau)\cos(2\omega s) + D(\tau)\sin(2\omega s). \quad (3.15)$$

Substituting (3.15) into equation (3.14) and following the same procedure as previously then five equations are obtained for A, C, D, Z and ω involving derivatives of A, C and D with respect to τ. These equations are rather messy and so attention is restricted to limit cycle solutions for which τ derivatives are zero:

$$\bar{a}_0 = 0 \Rightarrow 2\gamma Z^2 - 2Z + \gamma A^2 + \gamma C^2 + \gamma D^2 = 0, \quad (3.16a)$$

$$\bar{a}_1 = 0 \Rightarrow \omega^2 - ZD\omega + 2\gamma Z + \gamma C - 1 = 0, \quad (3.16b)$$

$$\bar{b}_1 = 0 \Rightarrow \tfrac{1}{8}A(4\mu - 4Z^2 - A^2 - 2C^2 - 2D^2 - 4ZC - 4\gamma D/\omega) = 0, \quad (3.16c)$$

$$\bar{a}_2 = 0 \Rightarrow 4\mu C - 4Z^2C - 2A^2C - C^3 - CD^2 - 2ZA^2 - 8\omega D - (4\gamma Z - 2)\,D/\omega = 0, \quad (3.16d)$$

$$\bar{b}_2 = 0 \Rightarrow 4\mu D - 4Z^2D - 2A^2D - D^3 - C^2D + 8\omega C + (4\gamma Z - 2)\,C/\omega + \gamma A^2/\omega = 0. \quad (3.16e)$$

Interpretation. 1. A solution of these five nonlinear algebraic equations can be derived for μ small and $|\gamma| \ll \sqrt{\mu}$:

$$Z_{\mathrm{L}} = 2\gamma\mu + O(\mu^2), \qquad A_{\mathrm{L}} = 2\sqrt{\mu} + O(\mu^{\frac{3}{2}}), \quad (3.17a, b)$$

$$C_{\mathrm{L}} = -\tfrac{2}{3}\gamma\mu + O(\mu^2), \quad D_{\mathrm{L}} = O(\mu^2), \quad \omega_{\mathrm{L}} = 1 - \tfrac{5}{3}\gamma\mu + O(\mu^2). \quad (3.17c–e)$$

If $\gamma = 0$ then the solution is to this order the same as that of the van der Pol equation. If γ is small the saddle point at $x = 1/\gamma$ is a long way from the origin and hence the limit cycle for small μ, as expected, is not affected by this unstable point.

The non-zero solution of equation (3.16) reveals that as μ approaches μ_{c} the orbit grows and tends to the saddle point at $x = 1/\gamma$. At $\mu = \mu_{\mathrm{c}}$ the solution is the known homoclinic orbit (see Merkin & Needham 1986), where the frequency of the orbit is zero, and the periodic solution no longer exists for $\mu > \mu_{\mathrm{c}}$. The three-dimensional system (3.16) undergoes a Hopf bifurcation which appears to coincide with the point of vertical inflection for the mean of oscillation in the parameter-response diagrams (cf. figure 16). This Hopf bifurcation indicates the presence of the homoclinic orbit, which manifests numerically as a rapid decrease in frequency and increase in the mean of oscillation towards the saddle point. Of course the series solution which requires a finite non-zero frequency cannot adequately represent the homoclinic orbit.

Figure 13 shows the periodic solution obtained numerically and via (3.16) for $(\mu, \gamma) = (0.01, 2.0)$ and $(0.036, 2.0)$ and the later case shows the need to include higher

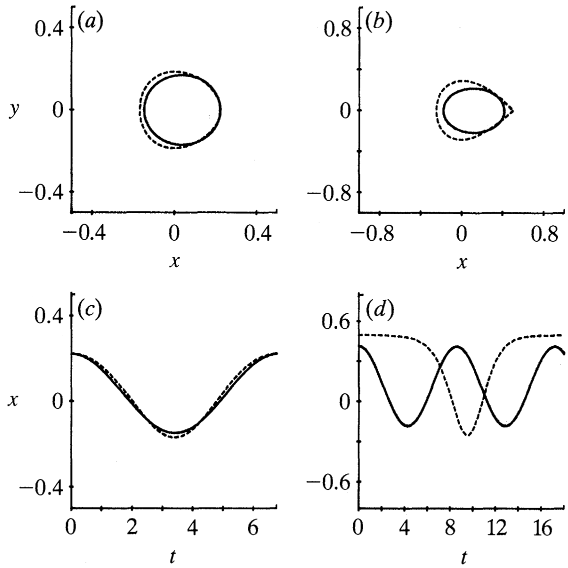

Figure 13. Comparison of a two harmonic approximation with numerical results of orbits at (*a*), (*c*) ($(\mu,\gamma) = (0.01, 2.00)$) and at (*b*), (*d*) ($(\mu,\gamma) = (0.036, 2.0)$) for the solution of equation (3.3). ——, 2THB; ----, numerical.

harmonics, particularly for the frequency equation, in the solution for $x(t)$ as the orbit approaches the saddle point at $x = 1/\gamma$.

2. The stability of the origin can be found by linearizing the amplitude evolution equations, i.e. (3.16*c*–*e*) with the τ derivatives, to obtain the stability matrix L

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\delta_A\\ \delta_C\\ \delta_D\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\mu & 0 & 0\\ 0 & \frac{1}{2}\mu & -6\\ 0 & 6 & \frac{1}{2}\mu\end{bmatrix}\begin{bmatrix}\delta_A\\ \delta_C\\ \delta_D\end{bmatrix}$$

and its eigenvalues $e_A = \frac{1}{2}\mu$, $e_{C,D} = \frac{1}{2}\mu \pm 6\mathrm{i}$. The origin is therefore stable (unstable) according as μ negative (positive).

The stability of the orbit growing out of the Hopf bifurcation at $(x,\mu) = (0,0)$ is found to be stable until the homoclinic orbit is reached.

(ii) $\mu = O(1)$

1. *The solution.* The vector field of the van der Pol equation with escape is asymmetric and hence even harmonics are retained in the expansion for $x(t)$ (cf. result in Appendix B). Results for the five harmonic expansions are shown in figure 14, for $(\mu,\gamma) = (0.036, 2.0)$ and $(0.142, -1.0)$. When $(\mu,\gamma) = (0.036, 2.0)$ the orbit approaches the saddle point at $x = 0.5$, and the oscillations spend a 'long time' near the saddle point as can be seen in the (x,t) plot. Also when $(\mu,\gamma) = (0.142, -1.0)$ in figure 14 the orbit is close to the saddle point but the agreement between analytical and numerical is still very good for a five harmonic expansion. A comparison of the amplitudes can be made with the relative frequency spectra, and in figure 15 with $(\mu,\gamma) = (0.14, -1.0)$ there is very good agreement.

Parameter-response diagrams are featured in figure 16 for $\gamma = 2.0$ and -1.0. The maximum and minimum extents of the oscillation are given by the maximum and

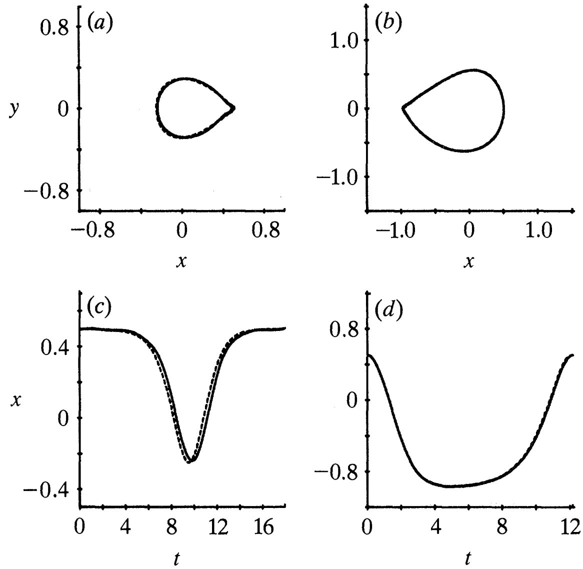

Figure 14. Comparison of a five harmonic approximation with numerical results of orbits at (*a*), (*c*) ($(\mu, \gamma) = (0.01, 2.00)$) and at (*b*), (*d*) ($(\mu, \gamma) = (0.036, 2.0)$) for the solution of equation (3.3). ——, 2THB; ----, numerical.

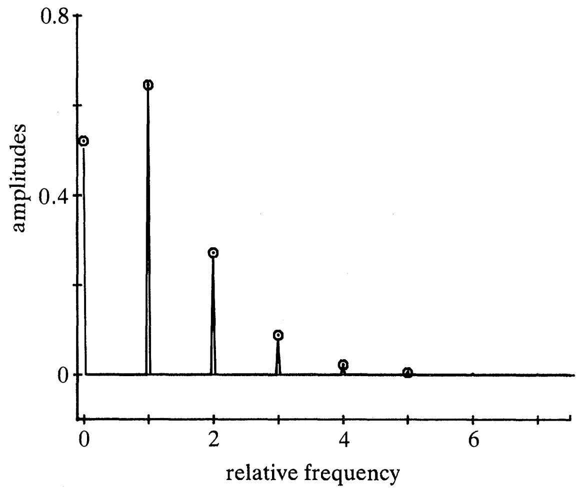

Figure 15. Numerical and analytical relative frequency spectra for $(\mu, \gamma) = (0.14, -1.0)$ in equation (3.3). ——, Numerical spectra; ⊙, 2THB amplitudes.

minimum response over a period. The mean of oscillation is determined (numerically or analytically) via

$$z = \bar{x} = \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s,$$

and all three are included in figure 16. The location of the homoclinic orbit is easily visualized where the maximum ($\gamma > 0$) or minimum ($\gamma < 0$) point of the oscillation touches $x = 1/\gamma$, and the solution undergoes a homoclinic bifurcation. Figure 16 also shows the μ variation of the frequency, depicting the rapid decrease in frequency at the critical value of μ, and an indication of the need for more harmonics.

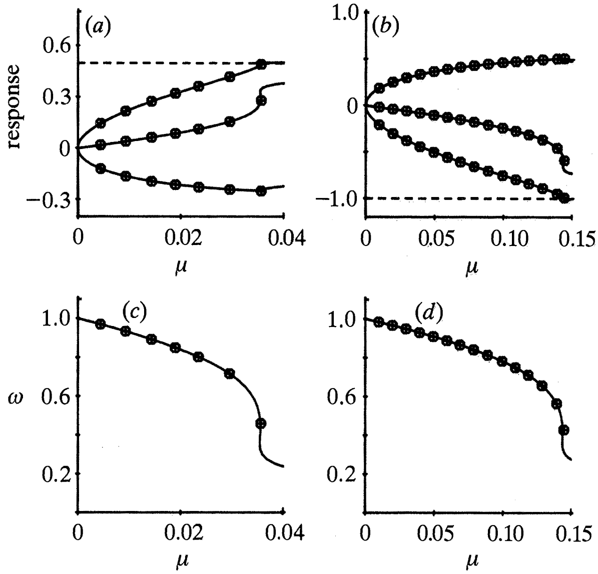

Figure 16. (*a*), (*b*) Numerical and analytical parameter-response curves for equation (3.3) at $\gamma = 2.0$ and -1.0. (*c*), (*d*) Numerical and analytical frequency variation with μ for equation (3.3). ——2THB; ⊕, numerical.

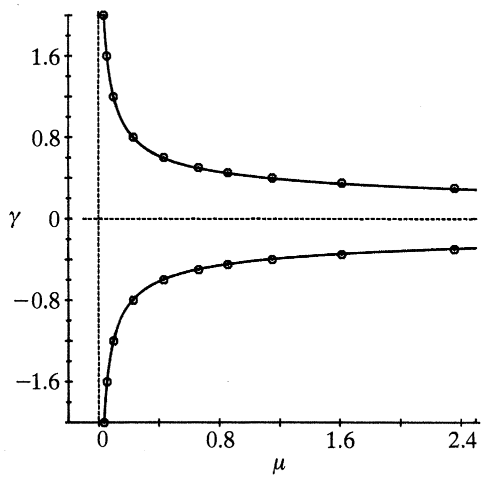

Figure 17. Control space (μ, γ) diagram showing the curve of homoclinic bifurcations which is the boundary of the limit cycle solutions. ——, 2THB; ⊙, numerical.

The resulting equations (3.7) for a three harmonic expansion of the oscillator given by equation (3.3) are given in Appendix C.

2. *The stability.* The stability is again determined by analysing the eigenvalues of the matrix L that occurs in equation (3.9). The analysis for the Hopf bifurcation at $\mu = 0$ is as discussed in §3*a* and this bifurcation is not γ dependent, however, a homoclinic bifurcation occurs which is γ dependent and is indicated by the change in stability of the system (3.8) which is manifested by a Hopf bifurcation. The locus of these bifurcation points (in (μ, γ) control space) is shown in figure 17, and a comparison is made with the numerical curve by the approach of Kaas-Petersen & Scott (1988).

4. Summary

The development of 2THB has been discussed through application to the autonomous van der Pol equation that has periodic solutions emanating from a Hopf bifurcation. The implementation of both a symbolic manipulator and path following routines has facilitated the evaluation and analysis of the amplitude evolution equations for the Fourier expansion of the solution, $x(t)$. In practice it is found that for $\mu = 1$ and 2, seven and nine harmonics were required respectively to represent the solution. However, when $\mu = 5$, 51 harmonics were necessary. This illustrates the restriction on the magnitude of μ according to the number of harmonics that can be handled by symbolic manipulation.

Application of this method to a further two autonomous oscillator equations, namely the modified van der Pol and the van der Pol with escape, permits a detailed discussion of local bifurcation points such as the Hopf and the saddle node. In the case of the van der Pol with escape, the approach of the periodic solutions to the homoclinic orbit is well resolved until the frequency equation breaks down as a result of the lack of higher harmonics in the expansion for $x(t)$.

The introduction of a small implicit parameter ϵ, permits fast and slow timescales associated with the phase and the amplitude variations respectively, whereby a stability argument can be incorporated. Often ϵ can be identified with an explicit small parameter present in the equation. The stability of the periodic solutions is determined by the collective stability of each term in the expansion. However, an approximation of the characteristic multiplier can be determined from the eigenvalues and eigenvectors of the stability matrix for the amplitude evolution equations, and in each case it shows good agreement with the numerical result.

In conclusion 2THB has the following three distinguishing features. Firstly, the method provides a simple representation for the periodic solutions as a finite series of harmonics throughout which the energy is distributed and which has a natural stability argument. Secondly, there is no need for an explicit small parameter. Thirdly, parameter-response curves and the location of bifurcation points are usually determined at less than a third of the computational cost of direct numerical integration.

We are indebted to Christian Kaas-Petersen for his guidance in the use of the path following package, PATH. We acknowledge many useful discussions with John Brindley. J.S. is supported by SERC, to which thanks and acknowledgements are also due. In addition we also thank the referees for their comments and in particular for pointing out the paper by Kuzmak.

Appendix A. Agreement between multiple scales and two timescale harmonic balance for a general autonomous second-order system

A general autonomous equation is considered for $\mu \ll 1$

$$\ddot{x} - \mu\dot{x} + x = f(x, \dot{x}) \tag{A 1}$$

with
$$f(0,0) = (\partial f/\partial x)_{(0,0)} = (\partial f/\partial \dot{x})_{(0,0)} = 0,$$

i.e. f is at least quadratic in x and $\dot{x}$. To apply multiple scales and two timescale harmonic balance $f(x, \dot{x})$ is expanded as a third-order Taylor series about the equilibrium position $(0, 0)$

$$f(x,\dot{x}) = \gamma_{00}\,x^2 + \gamma_{01}\,x\dot{x} + \gamma_{11}\,\dot{x}^2 + \gamma_{000}\,x^3 + \gamma_{001}\,x^2\dot{x} + \gamma_{011}\,x\dot{x}^2 + \gamma_{111}\,\dot{x}^3,$$

where $$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^{2-i-j}\,\partial \dot{x}^{i+j}}\right)_{(0,\,0)} \quad \text{and} \quad \gamma_{ijk} = \frac{1}{6}\left(\frac{\partial^3 f}{\partial x^{3-i-j-k}\,\partial \dot{x}^{i+j+k}}\right)_{(0,\,0)}.$$

Application of multiple scales reveals the existence of small amplitude periodic solutions to equation (A 1), of the form

$$x(t) = 2\left(\frac{-\mu}{\gamma_{00}\gamma_{01}+\gamma_{01}\gamma_{11}+\gamma_{001}+3\gamma_{111}}\right)^{\frac{1}{2}}\cos\left(1+O(\mu)\right)t+O(\mu) \tag{A 2}$$

providing that $\gamma_{00}\gamma_{01}+\gamma_{01}\gamma_{11}+\gamma_{001}+3\gamma_{111} \neq 0$.

By following the method of 2THB to derive solutions close to the limit cycle solution, equation (A 1) is written in the form

$$x''+\omega^2 x = (\omega^2-1)\,x+f(x,x')-2\epsilon\,\partial x'/\partial\tau+\epsilon\beta(x,x')\,\partial x/\partial\tau, \tag{A 3}$$

where $\beta(x,x') = \mu+\gamma_{01}x+2\gamma_{11}x'+\gamma_{001}x^2+2\gamma_{011}xx'+3\gamma_{111}x'^2$, and a two harmonic expansion for the periodic solutions and $O(\epsilon)$ close trajectories is sought

$$x(t) = Z+A_1(\tau)\cos\omega s+A_2(\tau)\cos 2\omega s+B_2(\tau)\sin 2\omega s,$$

where Z and the amplitudes, $A_1(\tau)$, $A_2(\tau)$ and $B_2(\tau)$, are determined as in §2 (equation (2.12)) by first deriving the following five equations

$$Z-\tfrac{1}{2}\gamma_{00}A_1^2-\tfrac{1}{2}\gamma_{11}A_1^2\omega^2+O(A_1^4)+C_{1Z}\,\mathrm{d}A_1/\mathrm{d}\tau+C_{2Z}\,\mathrm{d}A_2/\mathrm{d}\tau+C_{3Z}\,\mathrm{d}B_2/\mathrm{d}\tau = 0,$$

$$A_1(1-\omega^2-\gamma_{00}(2Z+A_2)-\tfrac{1}{2}\gamma_{01}B_2\,\omega-2\gamma_{11}A_2\,\omega^2-\tfrac{3}{4}\gamma_{000}A_1^2-\tfrac{1}{4}\gamma_{011}A_1^2\omega^2)$$
$$+O(A_1^4)+C_{1\omega}\frac{\mathrm{d}A_1}{\mathrm{d}\tau}+C_{2\omega}\frac{\mathrm{d}A_2}{\mathrm{d}\tau}+C_{3\omega}\frac{\mathrm{d}B_2}{\mathrm{d}\tau} = 0,$$

$$\epsilon\frac{\mathrm{d}A_1}{\mathrm{d}\tau}+C_{1A_1}\frac{\mathrm{d}A_1}{\mathrm{d}\tau}+C_{2A_1}\frac{\mathrm{d}A_2}{\mathrm{d}\tau}+C_{3A_1}\frac{\mathrm{d}B_2}{\mathrm{d}\tau}$$
$$= A_1(\mu\omega-\gamma_{00}B_2+\gamma_{01}\,\omega(Z+\tfrac{1}{2}A_2)-2\gamma_{11}B_2\,\omega^2+\tfrac{1}{4}\gamma_{001}A_1^2\omega+\tfrac{3}{4}\gamma_{111}A_1^2\omega^3)/2\omega+O(A_1^4),$$

$$\epsilon\frac{\mathrm{d}A_2}{\mathrm{d}\tau}+C_{1A_2}\frac{\mathrm{d}A_1}{\mathrm{d}\tau}+C_{2A_2}\frac{\mathrm{d}A_2}{\mathrm{d}\tau}+C_{3A_2}\frac{\mathrm{d}B_2}{\mathrm{d}\tau} = (B_2-4B_2\,\omega^2+2\mu A_2\,\omega+\tfrac{1}{2}\gamma_{01}A_1^2\,\omega)/2\omega+O(A_1^4),$$

$$\epsilon\frac{\mathrm{d}B_2}{\mathrm{d}\tau}+C_{1B_2}\frac{\mathrm{d}A_1}{\mathrm{d}\tau}+C_{2B_2}\frac{\mathrm{d}A_2}{\mathrm{d}\tau}+C_{3B_2}\frac{\mathrm{d}B_2}{\mathrm{d}\tau} = -(A_2-4A_2\,\omega^2-2\mu B_2\,\omega$$
$$-\tfrac{1}{2}\gamma_{00}A_1^2+\tfrac{1}{2}\gamma_{11}A_1^2\omega^2)/2\omega+O(A_1^4),$$

where C_{1Z}, C_{2Z}, C_{3Z}, $C_{1\omega}$, $C_{2\omega}$, $C_{3\omega}$, C_{1A_1}, C_{2A_1}, C_{3A_1}, C_{1A_2}, C_{2A_2}, C_{3A_2}, C_{1B_2}, C_{2B_2}, C_{3B_2} are all functions of Z, ω, A_1, A_2, B_2, μ and the γs. As the parameter μ is small the coefficients Z, A_1, A_2, and B_2 are ordered, so that $1 \gg A_1 \gg (Z, A_2, B_2)$. Limit cycle solutions of the above system are obtained by setting $\mathrm{d}A_1/\mathrm{d}\tau$, $\mathrm{d}A_2/\mathrm{d}\tau$ and $\mathrm{d}B_2/\mathrm{d}\tau$ to zero, and then we find, ω_{L}, Z_{L}, $A_{1\mathrm{L}}$, $A_{2\mathrm{L}}$ and $B_{2\mathrm{L}}$, as follows:

$$A_{1\mathrm{L}} = 2(-\mu/(\gamma_{00}\gamma_{01}+\gamma_{01}\gamma_{11}+\gamma_{001}+3\gamma_{111}))^{\frac{1}{2}}+O(\mu^{\frac{3}{2}}),$$
$$Z_{\mathrm{L}} = \tfrac{1}{2}(\gamma_{00}+\gamma_{11})A_{1\mathrm{L}}^2+O(\mu^2),$$
$$A_{2\mathrm{L}} = \tfrac{1}{6}(-\gamma_{00}+\gamma_{11})A_{1\mathrm{L}}^2+O(\mu^2),$$
$$B_{2\mathrm{L}} = \tfrac{1}{6}A_{1\mathrm{L}}^2\gamma_{01}+O(\mu^2),$$
$$\omega_{\mathrm{L}} = 1+\frac{1}{6}\left(\frac{10\gamma_{00}^2+\gamma_{01}^2+10\gamma_{00}\gamma_{11}+4\gamma_{11}^2+9\gamma_{000}+3\gamma_{011}}{\gamma_{00}\gamma_{01}+\gamma_{01}\gamma_{11}+\gamma_{001}+3\gamma_{111}}\right)\mu+O(\mu^2),$$

where the suffix L refers to values on the limit cycle solution. These results for the amplitude values on the periodic orbit are in agreement with those from multiple scales (equation (A 2)). Furthermore it can be shown, though it is a tedious task, that the amplitude evolution equations are also in agreement.

Appendix B

The dynamical system under investigation is given by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu), \quad \boldsymbol{x}, \boldsymbol{f} \in \mathbb{R}^2, \quad \mu \in \mathbb{R} \tag{B 1}$$

and is assumed to be antisymmetric. Given that $\boldsymbol{x} = (x_1, x_2) = (x, \dot{x})$ and $\boldsymbol{f} = (f_1, f_2)$, then a system is antisymmetric if

$$f_i(-x_1, -x_2, \mu) = -f_i(x_1, x_2, \mu) \quad \text{for} \quad i = 1, 2.$$

For example the van der Pol equation (3.1) is

$$\dot{x}_1 = x_2 = -(-x_2) = -f_1(-x_1, -x_2, \mu) = f_1(x_1, x_2, \mu),$$
$$\dot{x}_2 = -(x_1^2 - \mu)\,x_2 - x_1 = -((-x_1)^2 - \mu)(-x_2) - (-x_1) =$$
$$-f_2(-x_1, -x_2, \mu) = f_2(x_1, x_2, \mu).$$

It follows that for an antisymmetric oscillation it takes half the period to go from $(x_1, 0)$ to $(-x_1, 0)$. Therefore as the period of the oscillation is $2\pi/\omega$

$$x(t) = -x(t + \pi/\omega)$$

and hence

$$Z + \sum_{n=1}^{N} (A_n \cos(n\omega s) + B_n \sin(n\omega s)) = -Z - \sum_{n=1}^{N} (A_n \cos(n\omega s + n\pi) + B_n \sin(n\omega s + n\pi)).$$

This therefore implies that $Z = 0$ and the following equation holds

$$A_n(\cos(n\omega s) + \cos(n\omega s)\cos(n\pi) - \sin(n\omega s)\sin(n\pi))$$
$$+ B_n(\sin(n\omega s) + \sin(n\omega s)\cos(n\pi) + \cos(n\omega s)\sin(n\pi)) = 0. \tag{B 2}$$

Consequently if n is even then

$$A_n(2\cos(n\omega s)) + B_n(2\sin(n\omega s)) = 0 \;\forall s \Rightarrow A_n, \quad B_n = 0$$

and if n is odd the bracketed expressions of (B 2) are zero.

Therefore, if the autonomous system under investigation has an antisymmetric vector field, the above result indicate that the mean of oscillation and the even Fourier coefficients for the expansion of the solution $x(t)$ can be ignored.

Appendix C

Equations (3.8) are determined for the van der Pol equation with escape (3.3) when $x(t)$ is represented by a three harmonic expansion and the equations are given in full below:

$$x(t) = Z + A(\tau)\cos(\omega s) + C(\tau)\cos(2\omega s) + D(\tau)\sin(2\omega s) + E(\tau)\cos(3\omega s) + G(\tau)\sin(3\omega s),$$
$$\boldsymbol{H} = (A, C, D, E, G).$$

The matrix $M(Z, \boldsymbol{H}, \omega, \mu, \gamma)$ has the following components:

$M_{11} = 2(2ZD + AG - 4\omega)$, $M_{21} = 2(2ZG - CG + DE)$,
$M_{31} = 2(2ZA - 2ZE + CE + DG)$, $M_{41} = 2D(C - 2Z)$,
$M_{51} = 4ZC + A^2 - C^2 + D^2$, $M_{12} = 2(2ZA + 2ZE + 2AC + CE + DG)$,
$M_{22} = 4Z^2 + 2A^2 + 2AE + 3C^2 + D^2 + 2E^2 + 2G^2 - 4\mu$, $M_{32} = 2(AG + CD + 8\omega)$,
$M_{42} = 2(2ZA + AC + 2CE)$, $M_{52} = 2(AD + 2CG)$,
$M_{13} = 2(2ZG + 2AD + CG - DE)$, $M_{23} = 2(AG + CD - 8\omega)$,
$M_{33} = 4Z^2 + 2A^2 - 2AE + C^2 + 3D^2 + 2E^2 + 2G^2 - 4\mu$, $M_{43} = 2D(2E - A)$,
$M_{53} = 2(2ZA + AC + 2DG)$, $M_{14} = 4ZC + A^2 + 4AE + C^2 - D^2$,
$M_{24} = 2(2ZA + AC + 2CE)$,
$M_{34} = 2D(2E - A)$, $M_{44} = 4Z^2 + 2A + 2C^2 + 2D^2 + 3E^2 + G^2 - 4\mu$,
$M_{54} = 2(EG + 12\omega)$, $M_{15} = 2(2ZD + 2AG + CD)$,
$M_{25} = 2(AD + 2CG)$, $M_{35} = 2(2ZA + AC + 2DG)$,
$M_{45} = 2(EG - 12\omega)$, $M_{55} = 4Z^2 + 2A^2 + 2C^2 + 2D^2 + E^2 + 3G^2 - 4\mu$.

The vector $\boldsymbol{G}(Z, \boldsymbol{H}, \omega, \mu, \gamma)$ has the following components:

$$G_1 = 4\omega Z^2A + 4\omega ZAC + 4\omega ZCE + 4\omega ZDG + \omega A^3 + \omega A^2E + 2\omega AC^2 + 2\omega AD^2 + 2\omega AE^2 + 2\omega AG^2 - 4\omega\mu A + \omega C^2E + 2\omega CDG - \omega D^2E + 4\gamma AD + 4\gamma CG - 4\gamma DE,$$
$$G_2 = 2(8\omega^2C - 4\omega Z^2D - 4\omega ZAG - 2\omega A^2D - 2\omega ACG + 2\omega ADE - \omega C^2D - \omega D^3 - 2\omega DE^2 - 2\omega DG^2 + 4\omega\mu D + 4\gamma ZC + \gamma A^2 + 2\gamma AE - 2C),$$
$$G_3 = 2(8\omega^2D + 4\omega Z^2C + 2\omega ZA^2 + 4\omega ZAE + 2\omega A^2C + 2\omega ACE + 2\omega ADG + \omega C^3 + \omega CD^2 + 2\omega CE^2 + 2\omega CG^2 - 4\omega\mu C + 4\gamma ZD + 2\gamma AG - 2D),$$
$$G_4 = 36\omega^2E - 12\omega Z^2G - 12\omega ZAD - 6\omega A^2G - 6\omega ACD - 6\omega C^2G - 6\omega D^2G - 3\omega E^2G - 3\omega G^3 + 12\omega\mu G + 8\gamma ZE + 4\gamma AC - 4E,$$
$$G_5 = 36\omega^2G + 12\omega Z^2E + 12\omega ZAC + \omega A^3 + 6\omega A^2E + 3\omega AC^2 - 3\omega AD^2 + 6\omega C^2E + 6\omega D^2E + 3\omega E^3 + 3\omega EG^2 - 12\omega\mu E + 8\gamma ZG + 4\gamma AD - 4G.$$

The equation that determines Z_{L} is given by

$$\Xi = 2\gamma Z_{\mathrm{L}}^2 - 2Z_{\mathrm{L}} + \gamma(A_{\mathrm{L}}^2 + C_{\mathrm{L}}^2 + D_{\mathrm{L}}^2 + E_{\mathrm{L}}^2 + G_{\mathrm{L}}^2) = 0,$$
$$\Rightarrow Z_{\mathrm{L}} = (1/2\gamma)\,(1 \pm (1 - 2\gamma^2(A_{\mathrm{L}}^2 + C_{\mathrm{L}}^2 + D_{\mathrm{L}}^2 + E_{\mathrm{L}}^2 + G_{\mathrm{L}}^2))^{\frac{1}{2}}).$$

The equation that determines ω_{L} is given by

$$\Omega = -4\omega_{\mathrm{L}}^2 A_{\mathrm{L}} + 4\omega_{\mathrm{L}} Z_{\mathrm{L}} A_{\mathrm{L}} D_{\mathrm{L}} + 4\omega_{\mathrm{L}} Z_{\mathrm{L}} C_{\mathrm{L}} G_{\mathrm{L}} - 4\omega_{\mathrm{L}} Z_{\mathrm{L}} D_{\mathrm{L}} E_{\mathrm{L}} + \omega_{\mathrm{L}} A_{\mathrm{L}}^2 G_{\mathrm{L}} - \omega_{\mathrm{L}} C_{\mathrm{L}}^2 G_{\mathrm{L}} + 2\omega_{\mathrm{L}} C_{\mathrm{L}} D_{\mathrm{L}} E_{\mathrm{L}} + \omega_{\mathrm{L}} D_{\mathrm{L}}^2 G_{\mathrm{L}} - 8\gamma Z_{\mathrm{L}} A_{\mathrm{L}} - 4\gamma A_{\mathrm{L}} C_{\mathrm{L}} + 4A_{\mathrm{L}} - 4\gamma C_{\mathrm{L}} E_{\mathrm{L}} - 4\gamma D_{\mathrm{L}} G_{\mathrm{L}} = 0,$$
$$\Rightarrow \omega_{\mathrm{L}} = \{1 - \gamma A^{-1}(2ZA + AC + CE + DG) + (\tfrac{1}{8}A^{-1}(4ZAD + 4ZCG - 4ZDE + A^2G - C^2G + 2DE + D^2G))^2\}_{\mathrm{L}}^{\frac{1}{2}} + (\tfrac{1}{8}A^{-1}(4ZAD + 4ZCG - 4ZDE + A^2G - C^2G + 2DE + D^2G))_{\mathrm{L}}.$$

References

Bejarano, J. D. & Yuste, S. B. 1986 Construction of approximate analytical solutions to a new class of nonlinear oscillator equations. *J. Sound Vib.* **110**, 347–350.

Blaquière, A. 1951 Extension de la théorie de Nyquist au cas de caractéristiques nonlinéaires. *Compt. Rend.* **233**, 345.

Blaquière, A. 1966 *Nonlinear system analysis*. New York: Academic Press.

Bogoliubov, N. N. & Mitropolsky, Y. A. 1958 *Asymptotic methods in the theory of non-linear oscillations*. Moscow. (English translation, Gordon and Breach, New York, 1961.)

Carrier, G. F. 1953 Boundary layer problems in applied mechanics, §5. *Adv. appl. Mech.* **3**, 12–16.

Cochran, J. A. 1962 Problems in singular perturbation theory. Ph.D. thesis, Stanford University.

Dorodnicyn, A. A. 1947 Asymptotic solution of van der Pol's equation. *Appl. math. Mech.* **11**, 313–328.

Goldfarb, L. 1947 *Frequency response* (ed. X. Oldenburger) (English translation). (O nekotorykh nelinejnostjakh v sistemakh avtomaticeskogo regulirovanija. *Avtomatika i Telemekhanika* **8**, 349–383; K. voprosu o teorii vibracionnykh reguljatorov. *Avtomatika i Telemekhanika* **9**, 413–431.)

Hale, J. K. 1969 *Ordinary differential equations*. New York: Wiley-Interscience.

Kaas-Petersen, C. 1987 PATH: user's guide. Report, May 1987. Leeds University: Centre for Nonlinear Studies.

Kaas-Petersen, C. & Scott, S. K. 1988 Homoclinic orbits in a simple chemical model of an autocatalytic reaction. *Physica* D **32**, 461–470.

Kevorkian, J. 1961 The uniformly valid asymptotic representation of the solution of certain nonlinear differential equations. Ph.D. thesis, Caltech, Pasadena.

Kevorkian, J. 1966 The two variable expansion procedure for the approximate solution of certain nonlinear differential equations. *AMS Lectures Appl. Math.* **7**, 206–275.

Kochenburger, R. J. 1950 A frequency response method for analyzing and synthesizing contactor servomechanisms. *Trans. AIEE* **69**, 270–283.

Krylov, N. M. & Bogoliubov, N. N. 1934 *New methods of nonlinear mechanics applied to the study of the operation of electric oscillators*. Part I. Moscow: United Scientific and Technical Press.

Krylov, N. M. & Bogoliubov, M. N. 1937 *Introduction to nonlinear mechanics*. Acad. Sci. Ukr. SSR. (English translation, Princeton University Press, Princeton, 1947.)

Kuzmak, G. E. 1959 Asymptotic solutions of nonlinear second order differential equations with variable coefficients. *PMM* **23**, 515–526

Lindstedt, A. 1883 Differentialgleichungen der Störungstheorie. *Mem. Acad. Imp. St. Petersburg* **31**.

Lochak, P. & Meunier, C. 1988 *Multiphase averaging for classical systems*. New York: Springer-Verlag.

Loeb, J. 1951 Un criterium général de stabilité des servomécanismes sièges de phénomènes héréditaires. *Compt. Rend.* **233**, 344.

Mahony, J. J. 1962 An expansion method for singular perturbation problems. *J. Aust. math. Soc.* **2**, 440–463.

Margallo, J. G. & Bejarano, J. D. 1987 A generalization of the method of harmonic balance. *J. Sound Vib.* **116**, 591–595.

Merkin, J. H. & Needham, D. J. 1986 Infinite period bifurcations with an application to roll waves. *Acta Mechanica* **60**, 1–16.

Mickens, R. E. 1984 Comments on the method of harmonic balance. *J. Sound Vib.* **94**, 456–460.

Mickens, R. E. 1986 A generalization of the method of harmonic balance. *J. Sound Vib.* **111**, 515–518.

Morrison, J. A. 1966 Comparison of the modified method of averaging and two variable expansion procedure. *SIAM Rev.* **8**, 66–85.

Nayfeh, A. H. 1973 *Perturbation methods*. New York: Wiley-Interscience.

Oppelt, W. 1947 Locus curve method for regulators with friction. *J. Inst. Elec. Engng Lond.* Pt.II A, **94** (1 and 2).

Perko, L. M. 1969 Higher order averaging and related methods for perturbed periodic and quasi-periodic systems. *SIAM J. appl. Math.* **17**, 698–724.

Poincaré, H. 1892 *Les méthodes nouvelles de le mécanique céleste*, *I*. Paris: Gauthiers-Villars.

Rayna, G. 1987 *REDUCE: software for algebraic computation*. New York: Springer-Verlag Artificial Intelligence.

Sanders, J. A. & Verhulst, F. 1985 *Averaging methods in nonlinear dynamical systems*. New York: Springer-Verlag.

Savage, M. D. 1986 Internal Report, Department of Applied Mathematical Studies, University of Leeds.

Šiljak, D. D. 1969 *Nonlinear systems*. New York: Wiley.

Summers, J. L. 1991 Applications of the method of two time scale harmonic balance to problems of autonomous and periodically forced mechanical oscillators. Ph.D. thesis, University of Leeds.

Theodorchik, K. F. 1948 *Auto-oscillatory systems*. Moscow.

Tustin, A. 1947 The effects of backlash and of speed-development friction on the stability of closed-cycle control systems. *J. Inst. Elec. Engng Lond.* Pt.II A, **94** (1).

van der Pol, B. 1922 On a type of oscillation hysteresis in a simple triode generator. *Phil. Mag.* **43**, 177–93.

van der Pol, B. 1927 Forced oscillations in a circuit with nonlinear resistance (reception with reactive triode). *Lond., Edinb., Dublin phil. Mag. J. Sci.* **3**, 65–80.

Wilson, C. A. 1980 Perturbations and solar tables from Lacaille to Delambre: the rapprochement of observation and theory. *Arch. History exact Sci.* **22**, 53–188 (part I) and 189–304 (part II).

Received 13 May 1991; revised 20 February 1992; accepted 18 March 1992